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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3(a)	3(b)	3(c)	4	Exercise
3	4	3	3	6	6	25 Points

1.

$$\textbf{(1p)} \quad \dot{x} = \bar{v} \tag{1}$$
$$\textbf{(1p)} \quad \dot{y} = \bar{v}\theta \tag{2}$$
$$\textbf{(1p)} \quad \dot{\theta} = \frac{\bar{v}}{\ell}u, \quad u = \tan(\phi) \tag{3}$$

Accept the answers if given with v instead of $\bar{v}$.

2.

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ \theta_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta\bar{v} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ \theta_k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \delta\bar{v}/\ell \end{bmatrix} u_k + \begin{bmatrix} \delta\bar{v} \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

The LTI part (A matrix:**2p**, B matrix:**1p**), the constant term at the end (**1p**). OK, to only give matrices and the constant offset of the affine term.

3. (a) The tracking error states are derived as follows.

$$\tilde{x}_{k+1} = x_{k+1} - x^r_{k+1} = x_k + \delta\bar{v} - (x^r_k + \delta\bar{v}) = x_k - x^r_k = \tilde{x}_k \tag{5}$$
$$\tilde{y}_{k+1} = y_{k+1} - y^r_{k+1} = y_k + \delta\bar{v}\theta_k - \bar{y} = \tilde{y}_k + \delta\bar{v}\tilde{\theta}_k \tag{6}$$
$$\tilde{\theta}_{k+1} = \theta_{k+1} - \theta^r_{k+1} = \theta_k + \delta\bar{v}/\ell u_k - 0 = \tilde{\theta}_k + \delta\bar{v}/\ell u_k \tag{7}$$

The resulting error dynamics is then given by

$$\begin{bmatrix} \tilde{x}_{k+1} \\ \tilde{y}_{k+1} \\ \tilde{\theta}_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta\bar{v} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ \tilde{y}_k \\ \tilde{\theta}_k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \delta\bar{v}/\ell \end{bmatrix} u_k.$$

The A matrix with the error states **(1p)**, B matrix **(1p)**. Plugging the values in to show the result **(1p)**.

If the values with the equations are directly derived instead of giving the state space, and values are plugged in along the way showing clear progression, **(1p)** per each equation above.

(b) The A matrix has all its eigenvalues at 1 (i.e., $\rho(A) = 1$) **(1p)**, therefore the system is not asymptotically stable (stating that it is unstable by inspecting the $(\tilde{y}_k, \tilde{\theta}_k)$ subspace is also correct) **(2p)**.

(c)

$$\mathcal{C} = [B \ AB \ A^2B]\textbf{(1p)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 8 \\ 2 & 2 & 2 \end{bmatrix} \textbf{(1p)} \tag{8}$$

The matrix is rank deficient, the system is not controllable **(1p)**.

Reachable space spanned by $(0, 1, 0)$ and $(0, 0, 1)$, uncontrollable mode 1 (**1p**). The system is not stabilisable. (**1p** with justification).

This is caused by the small angle approximation (OK to mention as linearization approximation) and the constant velocity assumption for the input (**1p** for listing at least one reason).

4. Substituting the controller into the system leads to

$$\begin{bmatrix} \tilde{x}_{k+1} \\ \tilde{y}_{k+1} \\ \tilde{\theta}_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & p & 1-1=0 \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ \tilde{y}_k \\ \tilde{\theta}_k \end{bmatrix} \tag{9}$$

The controllable subspace spans the $\tilde{y}, \tilde{\theta}$ states, thus we ignore the pole at 1, which is uncontrollable. The controllable subspace is then

$$\begin{bmatrix} \tilde{y}_{k+1} \\ \tilde{\theta}_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ p & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_k \\ \tilde{\theta}_k \end{bmatrix}. \tag{10}$$

There should be an explanation of what the closed-loop dynamics is and how we can get the desired poles, e.g., writing the A matrix etc., similar to the description above. Explanation until this point is **(3p)** (**1p** for describing the closed loop dynamics, **2p** for describing how to get the poles using the gain p). We want both poles of the characteristic polynomial at 0.5.

$$(\mathbf{2p}) \quad \det(sI - A) = s(s-1) - 2p = (s-0.5)^2 \implies p = -0.125$$

Solving the characteristic polynomial for the 3x3 A matrix is also fine.

The error states $\tilde{y}$ and $\tilde{\theta}$ in this discrete-time model are asymptotically stable. The uncontrollable state $\tilde{x}_k$, on the other hand is stable, but not asymptotically stable. (**1p**)

Exercise 2

1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	2(c)	Exercise
3	3	4	3	4	4	4	25 Points

1. (a) The observability matrix is:

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \,(\mathbf{1\ p})$$

For O to have full rank, we need $b \neq 0$, all other parameters a, c, d are irrelevant for observability and therefore can be chosen arbitrarily **(2 p)**.

(b) The controllability matrix is:

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & c \end{bmatrix} \,(\mathbf{1\ p})$$

For P to have full rank, we need $c \neq 0$, all other parameters a, b, d are irrelevant for controllability and therefore can be chosen arbitrarily **(2 p)**.

(c) The closed loop system is:

$$\hat{A} = A + BK = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} k & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} k+1 & 0 \\ -2 & -1 \end{bmatrix} \,(\mathbf{1\ p})$$

For the closed loop system to be asymptotically stable we need that all eigenvalues are strictly smaller then 0 **(1 p)**. As $\hat{A}$ is a triangular matrix, the eigenvalues are the elements along the diagonal. Therefore, we require the following condition: $k + 1 < 0$, i.e. $k < -1$ **(1 p)**.
As for the fastest convergence rate, since the eigenvalue -1 is fixed, it will determine the fastest possible rate of convergence. This is achieved when the second eigenvalue is at least as fast, that is when $k + 1 \leq -1$, i.e. $k \leq -2$ **(1 p)**.

(d) From the system dynamics we directly obtain that $Kx(t) = ky(t)$ **(1 p)**, so the controller can be directly implemented using a output feedback **(2 p)**.
Note that the fact that for $b = 0$ the system is unobservable and therefore the observer will not converge (Part (a)) is irrelevant in this case.

2. (a) From the characteristic polynomial $\det(\lambda I - A)$, one can easilly derive that the eigenvalues of the system are -4, 2 and -8 **(2 p)**. As one of the eigenvalues (i.e. $\lambda = 2$) is strictly positive **(1 p)**, we can conclude that the system is not stable **(1 p)**.

(b) To determine whether the system is detectable, we have to perform the detectability test on the unstable eigenvalues and verify whether we achieve full rank. The detectability test is:

$$\text{rank}\left(\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}\right)$$

For $\lambda = 2$:

$$\begin{bmatrix} 2I - A \\ C \end{bmatrix} = \begin{bmatrix} 0 & -6 & 0 \\ 0 & 10 & 0 \\ -6 & 4 & 6 \\ 2 & 0 & 0 \end{bmatrix} \,(\mathbf{1\ p})$$

This matrix has clearly full rank (rank = 3) **(1 p)**.

As the detectability test has full rank with $\lambda = 2$, the system is detectable **(2 p)**.
Note that however the system is not observable as its observability matrix O defined as:

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 12 & 0 \\ 8 & -72 & 0 \end{bmatrix}$$

has clearly $\text{rank}(O) = 2$ (not full rank).

(c) To determine whether the system is stabilisable, we have to perform the stabilisability test on the unstable eigenvalues and verify whether we achieve full rank. The stabilisability test is:

$$\text{rank}\left(\begin{bmatrix}\lambda I - A & B\end{bmatrix}\right)$$

For $\lambda = 2$:

$$\begin{bmatrix} 2I - A & B \end{bmatrix} = \begin{bmatrix} 0 & -6 & 0 & 4 \\ 0 & 10 & 0 & 0 \\ -6 & 4 & 6 & 0 \end{bmatrix} \text{ } (\mathbf{1\ p})$$

This matrix has clearly full rank (rank = 3) **(1 p)**.

As the stabilisability test has full rank with $\lambda = 2$, the system is stabilisable **(2 p)**.
Note that however the system is not controllable as its controllability matrix P defined as:

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 4 & 8 & 16 \\ 0 & 0 & 0 \\ 0 & 24 & 144 \end{bmatrix}$$

has clearly $\text{rank}(P) = 2$ (not full rank).

Exercise 3

1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	Exercise
4	4	4	8	3	2	25 Points

1. (a) From the differential equation, we identify system states as $x_1(t) = z(t)$ and $x_2(t) = \dot{z}(t)$. Then we have

$$\begin{aligned}\dot{x}_1(t) &= \dot{z}(t) = x_2(t),\\ \dot{x}_2(t) &= \ddot{z}(t) = -a\dot{z}(t) - bz(t) + u(t) = -ax_2(t) - bx_1(t) + u(t)\end{aligned}$$

Using that $\dot{z}(t)$ is the measured system output, finally we can write the state-space form of the system as

$$\begin{aligned}\dot{x}(t) &= \underbrace{\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t),\\ y(t) &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} x(t) + \underbrace{0}_{D} u(t)\end{aligned}$$

(give **(1 p.)** for each matrix correctly written; **(4 p.)** in total)

(b) Since $z(0) = \dot{z}(0) = 0$, $x(0) = 0$ as well, i.e., the system has a zero initial condition (**(1 p.)**).

i. Solution 1:
From the state space form we can obtain the transfer function from input $u(t)$ to output $y(t)$ by using $G(s) = C(sI - A)^{-1}B$ (**(1 p.)**). Hence,

$$G(s) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ b & s+a \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\textbf{(1 p.)}} = \underbrace{\frac{s}{s^2 + as + b}}_{\textbf{(1 p.)}}.$$

ii. Solution 2
The transfer function from input $u(t)$ to output $y(t)$ can be obtained by using $y(t) = \dot{z}(t)$, the differential equation describing the system and solving the system of equations in the Laplace domain (**(1 p.)**). Hence,

$$\begin{aligned}s^2 z(s) - s\dot{z}(0) - z(0) + a(sz(s) - z(0)) + bz(s) &= u(s),\\ y(s) &= sz(s) - z(0).\end{aligned}$$

(**(1 p.)** for the correct Laplace domain expressions.) Then $G(s) = \frac{y(s)}{u(s)} = \frac{sz(s)}{u(s)} = \frac{s}{(s^2+as+b)}$ (**(1 p.)** for the correct expression).

(c) For $(a, b) = (4, 0)$ we have $G(s) = \frac{1}{s+4}$. We can use a final value theorem to obtain the impulse response $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s) = 0$. (**(1 p.)** for Laplace transform of an impulse being 1, and **(1 p.)** for the correct steady-state response)
For $(a, b) = (0, 4)$ we have $G(s) = \frac{s}{s^2+4}$. The final value theorem cannot be used in this case. However, the inverse Laplace transform can be used. Using

that the Laplace transform of the step is $\frac{1}{s}$ we have $\mathcal{L}^{-1}\{\frac{1}{s^2+4}\} = \frac{1}{2}\sin(2t)$. (**(1 p.)** for Laplace transform of an step, and **(1 p.)** for the correct steady-state response)

(d) For $a = 2$, $b = -3$, we have $\frac{s}{s^2+2s-3} = \frac{s}{(s+3)(s-1)}$. Hence the system has a stable ($s = -3$) and unstable ($s = 1$) pole and a zero at $s = 0$. Hence, for $\omega \in [0, 1)$rad/s the magnitude characteristic rises with 20dB/dec, for $\omega \in [1, 3)$rad/s it is constant, and finally, for $\omega \in [3, \infty)$rad/s it drops with 20 dB/dec. The phase characteristic rises at $\omega = 1$ rad/s because of the unstable pole and then drops because of the stable pole at $\omega = 3$ rad/s **(1 p.)**. Hence, the transfer function corresponds to the bode plot G_2. **(1 p.)**

For $a = 4$, $b = 3$, we have $\frac{s}{s^2+4s+3} = \frac{s}{(s+3)(s+1)}$. Hence the system has two stable poles ($s = -3$ and $s = -1$) and a zero at $s = 0$. The magnitude characteristic is the same as for the previous case. However, the phase characteristic drops twice for $\frac{\pi}{2}$ **(1 p.)**. Hence the correct pairing is with the transfer function G_1 **(1 p.)**.

For $a = 3$, $b = 0$ we have $\frac{s}{s^2+3s} = \frac{1}{(s+3)}$. Hence the magnitude characteristic is flat until $\omega = 3$ rad/s when it drops with 20 dB/dec and where the phase characteristic drops for $\frac{\pi}{2}$ **(1 p.)**. Hence, the correct pairing is with the transfer function G_4 **(1 p.)**.

For $a = 0.1$ and $b = 9$ we have $\frac{s}{s^2+0.1s+9}$. Hence, the system has a pair of stable conjugate-complex poles ($\omega_{1,2} = \frac{-0.1 \pm i\sqrt{35.99}}{2}$) that exhibit resonance in the magnitude characteristic and drop the phase characteristic for π at $\omega \approx 3$ rad/s **(1 p.)**. Hence, the correct pairing is with the transfer function G_3 **(1 p.)**.

2. (a) The zero of the system is -1 and the poles are $\frac{-3 \pm 3\sqrt{3}}{2}$. Hence, the system has neither zeros nor poles with non-negative real part. Consequently $P = 0$ (**1 p.**). Since $P = 0$, according to the Nyquist criterion, for stability we need $N = Z = 0$ encirclements of the point $-1/K$ (**1 p.**). When $K = -2$, then $-\frac{1}{K} = 0.5$. Since there are no encirclements of that point, the system is stable (**1 p.**).

(b) If the K keeps decreasing, it will enter the area in which the point $-\frac{1}{K}$ is encircled twice ($N = 2$, thus $Z = 2$) (**1 p.**), hence the system becomes unstable (**1 p.**).

Exercise 4

1	2	3	4(a)	4(b)	4(c)	4(d)	Exercise
3	4	4	4	2	4	4	25 Points

1. - Nonlinear: Due to x_1x_2 terms **[1pt]**
   - Time-invariant as dynamics do not explicitly depend on time **[1pt]**
   - Autonomous as there is no input and it is time invariant **[1pt]**

   Grading: **1 pt** for each question if answer correct and at least one correct justification.

2. For equilibria we require the state space equations $f(x) = \begin{bmatrix} -x_1 + bx_1x_2 \\ -2x_2 - bx_1x_2 + a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. **[1pt]**
   - One equilibrium at $\overline{x} = \begin{bmatrix} 0 \\ \frac{a}{2} \end{bmatrix}$ **[1pt]**
   - A second equilibrium at: $\tilde{x} = \begin{bmatrix} a - \frac{2}{b} \\ \frac{1}{b} \end{bmatrix}$ **[2pts]**

3. The Jacobian of the system is **[1pt]** :

$$A = \frac{\partial f}{\partial x}(x) = \begin{bmatrix} -1 + bx_2 & bx_1 \\ -bx_2 & -bx_1 - 2 \end{bmatrix}$$

   For $\overline{x} = \begin{bmatrix} 0 \\ \frac{a}{2} \end{bmatrix}$ the Jacobian is:

$$A = \begin{bmatrix} -1 + \frac{a}{2}b & 0 \\ -\frac{a}{2}b & -2 \end{bmatrix}$$

   The eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = \frac{1}{2}ab - 1$. Asymptotically stable if $ab < 2$ **[1pt]** and unstable if $ab > 2$. Note that the case $ab = 2$ is excluded by the assumption $a \neq \frac{2}{b}$.

   Equilibrium at $\tilde{x} = \begin{bmatrix} a - \frac{2}{b} \\ \frac{1}{b} \end{bmatrix}$:

$$A = \begin{bmatrix} 0 & ab - 2 \\ -1 & -ab \end{bmatrix}$$

The characteristic polynomial is: $\lambda^2 + \lambda ab + ab - 2 = 0$ **[1pt]**. The equilibrium is asymptotically stable if and only if all coefficients have the same sign, hence asymptotically stable if $ab > 2$ **[1pt]** and unstable if $ab < 2$.

4. The equilibrium is $\begin{bmatrix} 0 \\ \frac{a}{2} \end{bmatrix}$ and the Lyapunov function is:

$$V(x) = x_2 + x_1 - \frac{a}{2}\ln(x_2)$$

(a) We check that the set $S = \{x \mid x_1 \geq 0, x_2 \geq 0\}$ is invariant by analyzing the vector fields at the set boundary:

- For $x_1 = 0$: $\dot{x}_1 = 0, \dot{x}_2 = -2x_2 + \alpha$ **[1pt]**, so x_1 cannot become negative **[1pt]**.
- If $x_2 = 0$: $\dot{x}_1 = -x_1, \dot{x}_2 = a > 0$ **[1pt]**, thus the system always remains in the positive quadrant and S is an invariant set **[1pt]**.

(b) From the Jacobian we get directly, $\lambda_1 = -2$ and $\lambda_2 = 0$ **[1pt]**. The stability analysis is inconclusive due to the zero eigenvalue **[1pt]**.

(c) Analyze $\frac{d}{dt}V(x(t))$:

$$\begin{aligned} \frac{d}{dt}V(x(t)) &= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 - \frac{a}{2x_2} \end{bmatrix} \begin{bmatrix} -x_1 + bx_1x_2 \\ -2x_2 - bx_1x_2 + a \end{bmatrix} \quad \textbf{[1pt]} \\ &= -\frac{(2x_2 - a)^2}{2x_2} \quad \textbf{[1pt]}\text{for analysing signs.} \end{aligned}$$

Thus, $\forall x(t) \in S_c$ we have that $\frac{d}{dt}V(x(t)) \leq 0$ **[1pt]**. Consequently, $V(x(t))$ will remain smaller than c and the set S_c is invariant **[1pt]**.

(d) LaSalle's theorem states that all trajectories initialized in S_c converge to the largest invariant set in $M = \{x \in S_c | \nabla V(x) f(x) = 0\} = \{x \in S_c | x_2 = \frac{a}{2}\}$ **[2pts]**. To stay on M, x_2 must remain equal to $\frac{a}{2}$ which is only the case if $\dot{x}_2 = 0$. And since on M we have that $\dot{x}_2 = x_1$, it follows that we must have $x_1 = 0$, hence, we must be at the equilibrium $\overline{x}$. As $\overline{x}$ is the only invariant set in S_c, all trajectories starting in S_c will converge to $\overline{x}$ **[2pts]**.