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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2(a)	2(b)	2(c)	3(a)	3(b)	Exercise
3	4	4	4	5	5	25 Points

1. Applying the Laplace transform on (1) results in

$$s^2Y(s) + 15sY(s) + 20Y(s) = 25U(s). \quad \textbf{[2 p.]}$$

The resulting transfer function reads as

$$G(s) = \frac{25}{s^2 + 15s + 20}. \quad \textbf{[1 p.]}$$

2. (a) The transfer function from r to e can be obtained, for example, from the block diagram by noticing that $e(t) = r(t) - y(t)$ **[1 p.]**. Equivalently, in the Laplace domain, it holds that

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= \left(1 - \frac{KG(s)}{1 + KG(s)}\right) R(s) \\ &= \frac{1}{1 + KG(s)} R(s) \textbf{[2 p.]} \end{aligned}$$

where in the second step we exploited the equivalence $Y(s) = \frac{KG(s)}{1+KG(s)} R(s)$. Hence, the transfer from r to e reads as

$$T_{r \to e}(s) = E(s)/R(s) = \frac{1}{1 + KG(s)} = \frac{s^2 + 15s + 20}{s^2 + 15s + 20 + 25K}. \quad \textbf{[1 p.]}$$

(b) Recall that the Laplace transform of the unitary step input reference is $R(s) = \frac{1}{s}$ **[1 p.]**. Moreover, note that, for all $K > 0$, the poles of $T_{r \to e}$ have negative real part **[1 p.]**. Hence, by the final value theorem, the corresponding tracking error at steady state can be computed as

$$e_{ss} = \lim_{s \to 0} sT_{r \to e}(s)R(s) = \lim_{s \to 0} s \frac{1}{1 + KG(s)} \frac{1}{s} = \frac{1}{1 + KG(0)} = \frac{1}{1 + \frac{25}{20}K}. \quad \textbf{[1 p.]}$$

It follows that $e_{ss} < 0.1$ if and only if $K > 9 \cdot \frac{20}{25} = 7.2$ **[1 p.]**.

(c) The closed-loop transfer function is given by

$$\frac{KG(s)}{1 + KG(s)} = \frac{\frac{25K}{s^2+15s+20}}{1 + \frac{25K}{s^2+15s+20}} = \frac{25K}{s^2 + 15s + 20 + 25K}. \quad \textbf{[1 p., can also use denominator of (a)]}$$

The 2^{nd} order transfer function is of the form

$$\hat{G}(s) = \frac{C_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \textbf{[1 p.]}$$

where ζ is the damping ratio. Comparing the coefficients

$$\begin{aligned} \omega_n^2 &= 20 + 25K, \\ 2\zeta\omega_n &= 15, \end{aligned} \qquad \textbf{[1 p.]}$$

one gets that $\zeta = 0.5$ if and only if:

$$\begin{aligned} \frac{15}{2\sqrt{20+25K}} &= 0.5 \\ 20 + 25K &= \left(\frac{15}{2 \times 0.5}\right)^2 \\ K &= \frac{15^2 - 20}{25} = 8.2. \quad \textbf{[1 p.]} \end{aligned}$$

3. (a) The Nyquist stability criterion states that the number of closed-loop poles with positive real part is $Z = N + P$, where N is the number of clockwise encirclements of the point $(-1, 0)$ and P is the number of open-loop poles with positive real part **[1 p.]**.

   The poles of the open-loop system

$$L(s) = D(s)G(s) = 25K\frac{(s+1.5)}{(s+2)(s+12)(s^2+15s+20)}$$

   are $p_1 = -2$, $p_2 = -12$ and the roots of the polynomial $(s^2 + 15s + 20)$, i.e.,

$$p_{3,4} = \frac{-15 \pm \sqrt{15^2 - 80}}{2}. \quad \textbf{[1 p.]}$$

   Since all the poles $\{p_i\}_{i=1}^4$ have negative real part, then the open loop system $L(s)$ does not have unstable poles, hence $P = 0$ **[1 p.]**.

   Moreover, from the Nyquist plot in Figure 3, we note that there is no encirclement of the point $(-1, 0)$, i.e., $N = 0$ **[1 p.]**. Hence, we conclude that the closed-loop system is stable **[1 p.]**.

   (b) As K increases, the magnitude of $L(s)$ increases, thus, the curve in the Nyquist diagram expands **[1 p.]**. For K big enough, the curve will eventually encircle twice the point $(-1, 0)$ in clockwise direction, hence, $N = 2$ **[2 p.]**. It follows from the Nyquist stability criterion, that the closed-loop system becomes unstable when K is increased sufficiently **[2 p.]**.

Exercise 2

1	2	3	4	5(a)	5(b)	5(c)	Exercise
3	6	4	4	2	3	3	25 Points

1. Noticing that the matrix A is upper triangular, its eigenvalues are $\lambda_1 = -1$ **[1 p]** and $\lambda_2 = 1$ **[1 p]**. So the system is unstable. **[1 p]**

2. The observability matrix is

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} \gamma & 1 \\ -\gamma & -2\gamma + 1 \end{bmatrix}. \textbf{[1 p]}$$

For $\gamma = 1$ or $\gamma = 0$, the observability matrix Q loses rank and the system is not observable, so it is observable for $\gamma \notin \{0, 1\}$ **[2 p, 1 each]**.

The controllability matrix is

$$P = [B \ AB] = \begin{bmatrix} 1 & -1 - 2\beta \\ \beta & \beta \end{bmatrix}. \textbf{[1 p.]}$$

For $\beta = 0$ or $1 = -1 - 2\beta$, equivalently $\beta = -1$, the controllability matrix P loses rank and the system is not controllable, so it is controllable for $\beta \notin \{0, -1\}$ **[2 p., 1 each]**.

3. For the unstable eigenvalue $\lambda = 1$ we have

$$\lambda I - A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \textbf{[1 p.]}$$

For $\gamma = 1$ we have

$$\begin{bmatrix} C \\ \lambda I - A \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{bmatrix}$$

so the matrix has rank 1, and thus the unstable eigenvalue is not detectable. **[1 p.]**

For $\gamma = 0$ we have

$$\begin{bmatrix} C \\ \lambda I - A \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 0 & 0 \end{bmatrix}$$

so the matrix has rank 2, and thus the unstable eigenvalue is detectable. **[1 p.]**

Therefore, for both values the system is unobservable, but it is detectable for $\gamma = 0$, so we choose $\gamma = 0$. **[1 p.]**.

4. For the unstable eigenvalue $\lambda = 1$ we have

$$\lambda I - A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \textbf{[1 p.]}$$

For $\beta = 0$ we have

$$\begin{bmatrix} B & \lambda I - A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so the matrix has rank 1, and thus the unstable eigenvalue is not stabilizable. **[1 p.]**

For $\beta = -1$ we have

$$\begin{bmatrix} B & \lambda I - A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 0 \end{bmatrix}$$

so the matrix has rank 2, and thus the unstable eigenvalue is stabilizable. **[1 p.]**

Therefore, for both values the system is uncontrollable, but it is stabilizable for $\beta = -1$, so we choose $\beta = -1$. **[1 p.]**.

5. (a) For these values we have

$$\tilde{A}(k) = A + BkC\textbf{[1 p.]} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} + k \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2+k \\ 0 & 1+2k \end{bmatrix} \textbf{[1 p.]}$$

(b) Since the matrix is upper triangular **[1 p.]**, the system is asymptotically stable for $1 + 2k < 0$ **[1 p.]** , or equivalently $k < \frac{-1}{2}$. **[1 p.]**

(c) Since the eigenvalue $\lambda_1 = -1$ cannot be altered, that is the fastest convergence rate **[1 p.]**. To achieve it we need, $1 + 2k \leq -1$ **[1 p.]**, or equivalently $k \leq -1$. **[1 p.]**

Exercise 3

1	2	3 (a)	3(b)	3 (c)	Exercise
6	4	5	4	6	25 Points

1. For the given circuit the following holds $i(t) = C\frac{d}{dt}u_c(t)$ (**1p.**), and $L\frac{d}{dt}i(t) = u_L$ (**1p.** for),with $u_L = u(t) - R(i(t))i(t) - u_c(t)$ (**1p.**). By taking $x(t) := \begin{bmatrix} u_c(t) \\ i(t) \end{bmatrix}$ we have

$$\frac{d}{dt}x(t) = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R(i(t))}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t) = \begin{bmatrix} 0 & 1 \\ -1 & -R(i(t)) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \ (\mathbf{2p.})$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \ (\mathbf{1p.}).$$

(Give all points even if matrices are not separated as in the solution, but everything is written as one big matrix)

2. First we note that the system is not linear because $R(i(t))i(i)$ is, in general, non linear function of $i(t)$ (**1p.**). In order for system to be linear, $R(i(t))i(t)$ has to be linear function of the current $i(t)$ (**1p.**). Consequently, we have $R(i(t))i(t) = ai(t)$, where $a \in \mathbb{R}$, i.e., $R(i(t)) = a$ (**2p.**). Note that when we have $R(i(t))i(t) = ai(t) + b$, where $a, b \in \mathbb{R}$, i.e., $R(i(t)) = a + \frac{b}{i(t)}$ system is linear in input and output, but it has a constant offset matrix $D = \begin{bmatrix} 0 \\ b \end{bmatrix}$ hence, by definition, it is affine. (Give **2p.** if there is just conclusion without explanations.)

3. (a) We find the equilibria of a system by setting $\frac{d}{dt}x(t) \overset{!}{=} 0$. Hence, $i(t) = 0$ and $-u_c(t) + \frac{\sin(t)}{i(t)}i(t) + u(t) = 0 \ \Rightarrow \ u_c(t) = u(t)$. (**1p.**) The nonlinear part of a system $\sin(i(t))$ is linearized around $i(t) = 0$, hence we have $\sin(i(t)) \approx i(t)$ (**1p.**). Consequently, the linearised system has the following form

$$\frac{d}{dt}x(t) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \ (\mathbf{1p.}),$$

and the characteristic polynomial is $\det(sI - A) = s^2 - s + 1$ (**1p.**). Since the roots of the polynomial have positive real part, the system is unstable (**1p.**). (**Give also points if solved for** $u(t) = 0$)

(b) We begin by writing the energy function of the system which is given by $E(t) = \frac{1}{2}x(t)^\mathsf{T}Qx(t)$, where $Q = \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix} = I$ (**1p., also if not written in matrix form**). Consequently, $E(t) = \frac{1}{2}x(t)^\mathsf{T}x(t)$.
Then the power ,which is the instantaneous energy change, is given by

$$P(t) = \frac{d}{dt}E(t) = \frac{1}{2}\left(\frac{d}{dt}x(t)^\mathsf{T}x(t) + x(t)^\mathsf{T}\frac{d}{dt}x(t)\right) = \frac{1}{2}x(t)^\mathsf{T}\left(A^\mathsf{T} + A\right)x(t)$$
$$= x(t)^\mathsf{T}\begin{bmatrix} 0 & 0 \\ 0 & -R(i(t)) \end{bmatrix} x(t) = -R(i(t))i(t)^2. \ (\mathbf{1p.})$$

When magnitude $i(t)$ is small, $R(i(t)) \approx -1$. (**1p.**) Hence, the power of the system is increasing, meaning that $R(i(t))$ is not a passive element, as it is introducing additional energy into the system. (**1p**).

(c) The control law $u(t) = f(y(t)) + Ky(t)$ has to be such that it cancels out the non-linearity $-R(i(t))i(t) = \sin(i(t))$ (**1p.**). Hence, $f(y(t)) = -\sin(i(t))$ (**1p.**). Plugging in the the desired control law in the system dynamics, we obtain

$$\frac{d}{dt}x(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & K \end{bmatrix}}_{\tilde{A}} x(t) \text{ } (\mathbf{2p.}).$$

The characteristic polynomial of the closed loop system is $\det(sI - \tilde{A}) = s^2 - Ks + 1$ (**1p.**). The roots of the characteristic polynomial have a negative real part when all coefficients of $s^2 - Ks + 1$ have the same sign, hence in order for system to be asymptotically stable $K < 0$ has to hold (**1p.**). Note that when $K = 0$, the system is marginally stable, and when $K > 0$, the system is unstable.

Exercise 4

1	2	3(a)	3(b)	3(c)	Exercise
4	5	4	6	6	**25 Points**

1. Its an EQ points if and only if $\dot{x} = f(x) = 0$ **[1 p]**

   $\dot{x} = f(x) = -\alpha \nabla g(x) \implies f(\bar{x}) = -\alpha \nabla g(\bar{x}) = 0$ **[3 p]**.

2.
   - The system is LAS if $\nabla f(\bar{x})$ is negative definite (all eigenvalues have negative real parts) **[1 p]**
   - $\nabla f(\bar{x}) = A = -\alpha \nabla^2 g(\bar{x})$ **[1 p]**
   - Argue that $\nabla^2 g(\bar{x})$ positive definite implies that A is negative definite. **[3 p]** (For this section, give full points if the student understands that if the Eigenvalues of $\nabla^2 g(\bar{x})$ are positive then the Eigenvalues of $A = -\alpha \nabla^2 g(\bar{x})$ are negative) Example proof:
     Let γ_i be the Eigenvalues of A. Since $\nabla g^2(\bar{x})$ is symmetric the Eigenvalues are all real and $\gamma_i = -\alpha \lambda_i$ where λ_i are the eigenvalues of $\nabla^2 g(\bar{x})$. Since $\nabla^2 g(\bar{x})$ is positive definite, $\lambda_i > 0$ which implies that $\gamma_i = -\alpha \lambda_i < 0$.

3. (a)
   - $V(\bar{x}) = \|\bar{x} - \bar{x}\|^2 = 0$ **[1 p]**
   - $V(x) = 0 \implies \|\bar{x} - x\| = 0 \implies x = \bar{x}$ **[1 p]**
   - $V(x) \geq 0$ by the properties of norms **[1 p]**
   - $\lim_{x\to\infty} \|x - \bar{x}\|^2 \geq \lim_{k\to\infty} (\|x\| - \|\bar{x}\|)^2 = \infty$ **[1 p]** (Also accept that $\lim_{x\to\infty} x - \bar{x} = \infty$)

   (b) The steps are:

$$\dot{V}(x) = (\nabla V(x))^T f(x) \quad \textbf{[1 p]} \tag{1}$$
$$= (x - \bar{x})^T f(x) \quad \textbf{[2 p]} \tag{2}$$
$$= (x - \bar{x})^T (-\alpha \nabla g(x)) \quad \textbf{[2 p]} \tag{3}$$
$$- \alpha (x - \bar{x})^T \nabla g(x) \quad \textbf{[1 p]} \tag{4}$$

   (c) First step:

$$\dot{V}(x) = -\alpha (x - \bar{x})^T \nabla g(x) \tag{5}$$
$$= -\alpha (x - \bar{x})^T (\nabla g(x) - 0) \quad \textbf{[1 p]} \tag{6}$$
$$= -\alpha (x - \bar{x})^T (\nabla g(x) - \nabla g(\bar{x})) \quad \textbf{[1 p]} \tag{7}$$

   Specialize the condition from y to $\bar{x}$, i.e.,

$$(x - \bar{x})^T (\nabla g(x) - \nabla g(\bar{x})) \geq m \|x - \bar{x}\|^2 \quad \textbf{[1 p]} \tag{8}$$

Substitute the condition into the previous expression

$$\dot{V}(x) = -\alpha(x-\bar{x})^T(\nabla g(x) - \nabla g(\bar{x})) \tag{9}$$
$$\leq -\alpha m\|x-\bar{x}\|^2 \quad \textbf{[2 p]} \tag{10}$$

Note that $\dot{V}(x) < 0$ when $x \neq \bar{x}$ and $\dot{V}(\bar{x}) = 0$ and apply Lyapunov's Theorem to show GAS. **[1 p]**