Automatic Control Laboratory
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Signals and Systems II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	6	7	Exercise
4	4	4	5	2	3	3	25 Points

1.

$$m\ddot{x} = F_E \sin(\theta) + F_s \cos(\theta) \tag{1}$$

$$\ddot{x} = \frac{F_E \sin(\theta) + F_s \cos(\theta)}{m} \tag{2}$$

2.

$$\ddot{z} = (F_E \cos(\theta) - F_s \sin(\theta) - mg)/m \tag{3}$$

3.

$$\ddot{\theta} = J^{-1} l_2 F_s \tag{4}$$

4.

$$\ddot{x} = \frac{F_E\theta + F_s}{m} \tag{5}$$

$$\ddot{z} = \frac{F_E - F_s\theta - mg}{m} \tag{6}$$

$$\ddot{\theta} = \frac{l_2 F_s}{J} \tag{7}$$

5. The physical interpretation is that the rocket is hovering. The solution forces $\theta_e = 0$.

$$F_S = 0,\ F_E = mg,\ \theta_e = 0 \tag{8}$$

The sign of F_E should be consistent with what the student used for the sign of mg.

6.

$$\begin{bmatrix} \dot{x} \\ \dot{v}_x \\ \dot{z} \\ \dot{v}_z \\ \dot{\theta} \\ \dot{v}_\theta \end{bmatrix} = \begin{bmatrix} v_x \\ \frac{F_E\theta + F_s}{m} \\ v_z \\ \frac{F_E - F_s\theta - mg}{m} \\ v_\theta \\ \frac{l_2 F_s}{J} \end{bmatrix} \tag{9}$$

7.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{m} \\ 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & 0 \\ 0 & \frac{l_2}{J} \end{bmatrix} \tag{10}$$

A is nilpotent, in particular $A * A = 0$. Therefore, $A^n B = \mathbf{0}$ for $n \geq 2$. Therefore, there is no way that the controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^5B \end{bmatrix} \tag{11}$$

can be full-rank. Therefore, the system is not controllable.

The correct C matrices are

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{12}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{13}$$

$\text{rank}\mathcal{C}_O(A, C_1) = 5$, so it is not observable. $\text{rank}\mathcal{C}_O(A, C_2) = 6$, so it is observable.

Exercise 2

1	2	3	4	5	Exercise
4	3	3	6	9	25 Points

1. The controllability matrix is given as

$$P = \begin{bmatrix} 1 & a_1 + a_2 \\ 1 & a_3 + a_4 \end{bmatrix}.$$

The system is controllable when P has full rank. P has full rank when $a_1 + a_2 \neq a_3 + a_4$. Hence, the system is controllable for all values of a_1, a_2, a_3, a_4 such that $a_1 + a_2 \neq a_3 + a_4$. The observability matrix is given as

$$Q = \begin{bmatrix} 0 & 1 \\ a_3 & a_4 \end{bmatrix}.$$

The system is observable when Q has full rank. Q has full rank when $a_3 \neq 0$. Hence, the system is observable for all values of a_1, a_2, a_3, a_4 such that $a_3 \neq 0$.

2. **Case 1:** Since the system is unobservable there exists no state $[x_1, x_2]^\top \in \mathbb{R}^2$ that is completely observable as x_1 can never be reconstructed from output measurements. Hence the set is empty.
Case 2: The state x_2 is observable since it is measured. However, the state x_1 is unobservable since it can never be reconstructed from output measurements.
Case 3: The set of unobservable points coincides with the null space of the observability matrix. Substituting the values for a_1, a_2, a_3, a_4 yields

$$Q = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}.$$

Thus, the set of unobservable points are given as $[x_1, 0]^\top \in \mathbb{R}^2$. Hence, the set of observable states are given as $[x_1, x_2]^\top \in \mathbb{R}^2$ such that $x_2 \neq 0$.

3. The set of reachable states is given by the image of the controllability matrix P. Substituting the values for a_1, a_2, a_3, a_4 yields

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

Hence, $X = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_1 = x_2 \right\}$.

4. The matrix A is diagonalizable. Hence, it can be written as

$$A = T\Lambda T^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Thus,

$$e^{At} = \begin{bmatrix} e^t & -e^t + e^{2t} \\ 0 & e^{2t} \end{bmatrix}.$$

5. Since the point $[1,1]^\top$ is in the set X from the previous problem, then it is possible to design a controller to steer the system from $x(0) = [0,0]^\top$ to $x(1) = [1,1]^\top$. Note that the solution to the system $\dot{x}(t) = Ax(t) + Bu(t)$ at time $t = 1$ is given by

$$x(1) = e^{At}x(0) + \int_0^1 e^{A(1-\tau)}Bu(\tau)d\tau.$$

Substituting in the values for A, B, $x(0)$ and the matrix exponential from the previous problem yields

$$x(1) = \int_0^1 \begin{bmatrix} e^{1-\tau} & -e^{1-\tau} + e^{2-2\tau} \\ 0 & e^{2-2\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(\tau)d\tau = \int_0^1 \begin{bmatrix} e^{2-2\tau} \\ e^{2-2\tau} \end{bmatrix} u(\tau)d\tau.$$

Setting $u(t) = e^{-2+2t}$ for $t \in [0,1]$ yields $x(1) = [1,1]^\top$.

Exercise 3

1	2	3	4	5	6	Exercise
4	5	4	4	5	3	25 Points

1. To find the equilibria, we need to impose

$$rx - x^3 = 0,$$

that results in two solutions: $x = 0$ and $x^2 = r$.

$r < 0$: In this case, the only possible equilibrium is $x = 0$, because all equilibria must be real valued.

$r = 0$: Also in this case the only equilibrium is $x = 0$.

$r > 0$: There are three equilibrium points, that is $x = 0$ and $x = \pm\sqrt{r}$

2. The linearized dynamics around the point $\bar{x}$ is

$$\dot{x} = \left[\frac{\partial f}{\partial x}\right]_{x=\bar{x}} x = \left(r - 3\bar{x}^2\right) x.$$

$r < 0$: The linearized dynamics around $x = 0$ is $\dot{x} = rx$. Therefore, we can conclude that $x = 0$ is an asymptotically stable equilibrium.

$r = 0$: The linearized dynamics around $x = 0$ is $\dot{x} = 0$. Nothing can be concluded by means of the linearization method.

$r > 0$: The linearized dynamics around $x = 0$ is $\dot{x} = rx$, so in this case $x = 0$ is an unstable equilibrium. On the other hand, the linearized dynamics around $x = \sqrt{r}$ is $\dot{x} = -2rx$, and therefore $x = \sqrt{r}$ is an asymptotically stable equilibrium. The same conclusion holds for $x = -\sqrt{r}$.

3. Consider the following quadratic candidate Lyapunov function

$$V(x) = \frac{1}{2}x^2 > 0 \quad \forall x \neq 0. \tag{14}$$

The Lie derivative of (14) along system trajectories is

$$\dot{V}(x) = x\dot{x} = rx^2 - x^4,$$

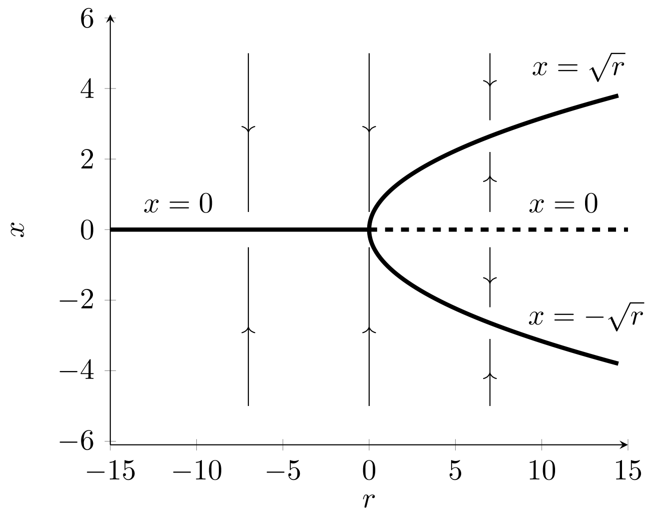
that, for $r = 0$, becomes $\dot{V}(x) = -x^4$, which is negative definite in $x = 0$. Therefore, we conclude that $x = 0$ is an asymptotically stable equilibrium.

4. **$r < 0$**: We can use again the positive definite Lyapunov function (14). Since $||x|| \to \infty \Rightarrow V(x) \to \infty$, function (14) is radially unbounded. Moreover, $\dot{V}(x) = rx^2 - x^4 < 0 \; \forall x \neq 0$, that allows us to conclude that $x = 0$ is a globally asymptotically stable equilibrium.

$r = 0$: With a similar discussion, it is easy to show that $x = 0$ is a globally asymptotically stable equilibrium.

$r > 0$: In this case, the asymptotically stable equilibria $x = \pm\sqrt{r}$ cannot be globally asymptotically stable. In fact, the presence of multiple attractors (or repulsors) clearly does not permit the existence of a globally stable equilibrium.

5. Pitchfork bifurcation. Solid lines represent stable equilibria, while dashed lines unstable equilibria. Arrows shows system trajectories for some fixed $r < 0$, $r = 0$ and $r > 0$.



6.

- $\dot{x}(t) = rx(t) + x^3(t)$ | a | b | c | ~~d~~ | none |
- $\dot{x}(t) = -rx(t) + x^3(t)$ | a | ~~b~~ | c | d | none |
- $\dot{x}(t) = -rx(t) - x^3(t)$ | ~~a~~ | b | c | d | none |

Exercise 4

1	2	3	4	5	Exercise
3	6	6	4	6	25 Points

1. The transfer function is given by $G(s) = C(sI - A)^{-1}B$ **(1 pt.)**, for the system Σ_1 this results in

$$\begin{aligned} G_1(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 5 \\ -1 & s-a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{(s+2)(s-a)+5} \begin{bmatrix} s-a & -5 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ &= \frac{1}{(s+2)(s-a)+5}, \qquad \textbf{(2 pts.)} \\ &= \frac{1}{s^2 + (2-a)s + 5 - 2a}. \end{aligned}$$

2. The natural frequency $\omega_n = \sqrt{5-2a}$ **(1 pt.)** and the damping factor $\zeta = \frac{2-a}{2\sqrt{5-2a}}$ **(1 pt.)**. The range of a for critically damped

$$\begin{aligned} &\zeta = \frac{2-a}{2\sqrt{5-2a}} = 1 \qquad \textbf{(1 pt.)} \\ &\Rightarrow a^2 - 4a + 4 = 20 - 8a \\ &\qquad \Rightarrow a = -2 \pm \sqrt{20}. \qquad \textbf{(1 pt.)} \end{aligned}$$

For $a = 2$, $G(s)$ has a pole at $s = \pm j\omega$ which means that for $y_1(t)$ is unbounded for $u_1(t) = 3\sin(t)$ **(2 pts.)**.

3. The closed loop transfer function is given by

$$\frac{KG_1(s)}{1 + KG_1(s)} = \frac{K}{s^2 + 12s + 25 + K}.$$

The poles of this transfer function are given by $-6 \pm \sqrt{11 - K}$. The poles have negative real part when $K > -25$.

4. The Laplace transform of $y_0(t)$ is $Y_0(s) = \frac{1}{s^2}$**(1 pt.)**, hence, $F(s) = \frac{1}{s}$ **(1 pt.)** (if they actually show how to compute $Y_0(s)$ they get **(2 pts.)**).

5. The signal $y_b(t)$ can be written as

$$y_b(t) = y_0(t) - 2y_0(t-1) + y_0(t-2) \qquad \textbf{(3 pts.)}.$$

Using the time-shift property of the Laplace transform we can write $Y_b(s)$ as

$$Y_b(s) = \frac{1 - 2e^{-s} + e^{-2s}}{s^2} \qquad \textbf{(3 pts.)}.$$

Using: $\mathcal{L}\{y_0(t-c)u_0(t-c)\} = e^{-cs}Y_0(s)$