Automatic Control Laboratory
ETH Zurich
Prof. J. Lygeros

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Signals and Systems II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	Exercise
8	9	8	25 Points

1. Substituting the expressions for y_k and $\hat{y}_k$ into the first line of the observer state update equation, and then subtracting the true system's state update equation yields

$$e_{k+1} = \hat{x}_{k+1} - x_{k+1} = (A - LC)e_k \,. \tag{1}$$

Examining the error dynamics in (1), for every $x_0 \in \mathbb{R}^n$, $\hat{x}_k \to x_k$ as k tends to infinity iff $|\lambda_i\{A - LC\}| < 1$ where λ_i are eigenvalues of matrix $A - LC$.

2. The observability matrix is $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -7 \\ 1 & -7 & 34 \end{bmatrix}$. Since $rank(\mathcal{O}) = 3$ then the system is **observable**.

The controllability matrix is $\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & -9 & 63 \\ 0 & -15 & 96 \\ 1 & -7 & 34 \end{bmatrix}$. Since $rank(\mathcal{C}) = 3$ then the system is also **controllable**.

When $u_k \equiv 0$, the system simplifies to $x_{k+1} = Ax_k$. The three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A are the roots of the characteristic polynomial

$$\det(\lambda I - A) = 0 \,,$$

which is $\lambda^3 + 7\lambda^2 + 15\lambda + 9 = 0$. This factorises to $(\lambda + 1)(\lambda + 3)^2 = 0$, whose roots are $(-1, -3, -3)$. Since two of these are strictly outside the unit circle, the system is **unstable**.

3. By inspection of the dimension of matrices C and A, $dim(L) = 3 \times 1$.

Now suppose $L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$; we wish to choose the entries of the observer gain matrix L such that $\lambda_i\{A - LC\} = +0.5$ for $i = 1, 2, 3$.

$$A - LC = \begin{bmatrix} 0 & 0 & -9 - l_1 \\ 1 & 0 & -15 - l_2 \\ 0 & 1 & -7 - l_3 \end{bmatrix} \Rightarrow \lambda^3 + (7 + l_3)\lambda^2 + (15 + l_2)\lambda + (9 + l_1) = (\lambda - 0.5)^3 \tag{2}$$

Where λ is the eigenvalue of the matrix $A - LC$.
Equating the polynomial coefficients in (2) gives

$$9 + l_1 = -0.125 \,, \quad 15 + l_2 = 0.75 \,, \quad 7 + l_3 = -1.5 \,.$$

This leads to $l_1 = -9.125$, $l_2 = -14.25$, and $l_3 = -8.5$.

Exercise 2

1	2	3	4	5	Exercise
5	4	4	8	4	25 Points

1. The flow balance equation for the tank takes the following form

$$\dot{m}(t) = q_{\text{in}} - q_{\text{out}}(t) = q_{\text{in}} - \rho A_v \sqrt{2gh(t)}\, v(t).$$

The water mass change can be written as

$$\dot{m}(t) = \rho \dot{V}(t) = \rho A_t \dot{h}(t).$$

Thus, we have

$$\dot{h}(t) = \frac{q_{\text{in}} - \rho A_v \sqrt{2gh(t)}\, v(t)}{\rho A_t}.$$

Dynamics of the valve position can be obtained from $G(s)$, *i.e.* we obtain

$$\dot{v}(t) = -30v(t) + 2u(t).$$

2. The system is
   (a) *nonlinear* because the dynamics equation f_1 is nonlinear in $h(t)$,
   (b) *time-invariant* because f_1 and f_2 are not direct functions of time.

3. In order to maintain a constant water height, we must have $\dot{m}(t) = 0$, and thus we obtain

$$\bar{v} = \frac{q_{\text{in}}}{\rho A_v \sqrt{2g\bar{h}}}.$$

Since the valve position is kept constant, we have $\dot{v}(t) = 0$, and thus

$$\bar{u} = 15\bar{v}.$$

4. The partial derivatives required for computing A and B are as follows:

$$\left.\frac{\partial f_1(h,v,u)}{\partial h}\right|_{\substack{h=\bar{h}\\ v=\bar{v}\\ u=\bar{u}}} = \frac{-A_v\sqrt{2g\bar{h}}}{A_t}\frac{\bar{v}}{2\bar{h}}$$

$$\left.\frac{\partial f_1(h,v,u)}{\partial v}\right|_{\substack{h=\bar{h}\\ v=\bar{v}\\ u=\bar{u}}} = \frac{-A_v\sqrt{2g\bar{h}}}{A_t}$$

$$\left.\frac{\partial f_1(h,v,u)}{\partial u}\right|_{\substack{h=\bar{h}\\ v=\bar{v}\\ u=\bar{u}}} = 0$$

$$\left.\frac{\partial f_2(h,v,u)}{\partial h}\right|_{\substack{h=\bar{h}\\ v=\bar{v}\\ u=\bar{u}}} = 0$$

$$\left.\frac{\partial f_2(h,v,u)}{\partial v}\right|_{\substack{h=\bar{h}\\ v=\bar{v}\\ u=\bar{u}}} = -30$$

$$\left.\frac{\partial f_2(h,v,u)}{\partial u}\right|_{\substack{h=\bar{h}\\ v=\bar{v}\\ u=\bar{u}}} = 2.$$

Therefore, we obtain

$$A = \begin{bmatrix} \frac{-A_v\sqrt{2g\bar{h}}}{A_t}\frac{\bar{v}}{2\bar{h}} & \frac{-A_v\sqrt{2g\bar{h}}}{A_t} \\ 0 & -30 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

5. The system is stable because its eigenvalues have negative real parts (we can read them from the diagonal of A). This means that the nonlinear system is locally asymptotically stable around this equilibrium.

Exercise 3

1	2	3	4	5	Exercise
3	3	7	8	4	25 Points

1. Let $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$. Then

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -cx_2(t) - \sin(x_1(t)) \end{pmatrix}.$$

Hence,

$$f(x(t)) = \begin{pmatrix} f_1(x(t)) \\ f_2(x(t)) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -cx_2(t) - \sin(x_1(t)) \end{pmatrix}$$

2. An equilibrium point $x = (x_1, x_2)$ satisfies $f(x) = 0$. Hence, an equilibrium point must satisfy

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_2 \\ -cx_2 - \sin(x_1) \end{pmatrix}$$

This implies that $x_2 = 0$. Thus, x_1 must satisfy $\sin(x_1) = 0$. Thus, the equilibrium points are of the form $x = (k\pi, 0)$ for any $k \in \mathbb{Z}$.

3. We denote by $\tilde{x}$ the state of the system linearized about equilibrium point $x = (0,0)$. Using the fact that $\sin(a) \approx a$ for small a, the linearization is given as

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{pmatrix} &= \begin{pmatrix} \tilde{x}_2(t) \\ -c\tilde{x}_2(t) - \tilde{x}_1(t)) \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}}_{A} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix}. \end{aligned}$$

The eigenvalues of the matrix A are given by the solutions of $\lambda^2 + c\lambda + 1 = 0$. Thus,

$$\begin{aligned} \lambda_1 &= \frac{-c + \sqrt{c^2 - 4}}{2} \\ \lambda_2 &= \frac{-c - \sqrt{c^2 - 4}}{2}. \end{aligned}$$

Case 1 ($c < 0$): Since $c < 0$ then $-c > 0$. For $-2 < c < 0$, $\mathrm{Re}(\sqrt{c^2 - 4}) = 0$. Hence, $\mathrm{Re}(\lambda_i) > 0$ for $i \in \{1, 2\}$. For $c \leq -2$, $\sqrt{c^2 - 4} \geq 0$. Hence, $\mathrm{Re}(\lambda_1) > 0$. Thus, for all $c < 0$, there exists an eigenvalue of A with positive real part proving that the linearization around the equilibrium point $x = (0,0)$ is unstable. Thus, the equilibrium point $x = (0,0)$ is unstable for the nonlinear system.

Case 2 ($c > 0$): For $0 < c < 2$ we have $\mathrm{Re}(\sqrt{c^2 - 4}) = 0$. Hence, $\mathrm{Re}(\lambda_i) < 0$ for $i \in \{1, 2\}$. For $c \geq 2$ we clearly have $\mathrm{Re}(\lambda_2) < 0$. Lastly, since $c > \sqrt{c^2 - 4}$ for $c \geq 2$, we also have $\mathrm{Re}(\lambda_1) < 0$. Thus, the equilibrium point $x = (0,0)$ is locally asymptotically stable for the nonlinear system.

4. Let $V(x_1, x_2) = 1 - \cos(x_1) + \frac{x_2^2}{2}$. Then V is a Lyapunov function in the set $S = (-\pi, \pi) \times \mathbb{R}$. Indeed, we have that $V(0, 0) = 0$. Also, $V(x_1, x_2) > 0$ for all $(x_1, x_2) \in S \setminus \{(0, 0)\}$. Lastly,

$$\frac{\mathrm{d}}{\mathrm{dt}} V(x_1(t), x_2(t)) = \sin(x_1(t))x_2(t) + x_2(t)(-cx_2(t) - \sin(x_1(t))).$$

Substituting $c = 0$ gives

$$\frac{\mathrm{d}}{\mathrm{dt}} V(x_1(t), x_2(t)) = 0.$$

Thus, the equilibrium point is stable for $c = 0$.

5. The equilibrium point cannot be globally asymptotically stable since the system has more than one equilibrium point.

Exercise 4

1	2	3	Exercise
5	10	10	25 Points

1. Applying the Laplace transform results in

$$s^2y(s) + s(2-\alpha)y(s) - 2\alpha y(s) = su(s) - u(s),$$
$$(s^2 + (2-\alpha)s - 2\alpha)y(s) = (s-1)u(s),$$

and we obtain the transfer function

$$G(s) = \frac{s-1}{s^2 + (2-\alpha)s - 2\alpha}.$$

2. Substituting $\omega = 0$ into $|G(j\omega)|$ results in

$$|G(j\omega)| = \frac{|j\omega - 1|}{|(j\omega)^2 + (2-\alpha)j\omega - 2\alpha)|},$$
$$|G(0)| = \frac{1}{2|\alpha|},$$

and for $\alpha = -0.01$ we obtain $|G(0)| = 50$, or $|G(0)| = 20\log(50)$ dB. It can be seen that both the magnitude and phase in the Bode plot in Figure 1 are monotone decreasing, i.e., the Small gain theorem can be applied. The Small gain theorem requires that $|KG(j\omega)| < 1$ for all ω. Next, we can use the fact that $|G(j\omega)| \leq |G(0)| = 50$ to obtain the condition $|K50| < 1$ and $|K| < \frac{1}{50}$. Alternatively, we can obtain the same result using properties of the Bode diagram. Because $20\log(|KG(j\omega)|) = 20\log(K) + 20\log(|G(j\omega)|)$, the controller Σ_2 shifts the gain plot by $20\log(K)$ dB. Considering that the maximum gain $|G(0)| = 20\log(50)$ dB occurs at $\omega = 0$, the feedback interconnection is stable for $|K| < \frac{1}{50}$.

3. The open loop system Σ has two poles $s_1 = -2$ and $s_2 = 10$, i.e., the number of open loop poles with positive real part is $P = 1$. The Nyquist stability criterion states that the number of closed loop poles with positive real part is $N = Z - P$, where N is the number of clockwise encirclements of the point $(\frac{1}{K}, 0)$ and Z is the number of closed loop poles with positive real part. For a stable system we require $Z = 0$ and therefore $N = -1$, i.e., one counter clockwise encirclement of the point $(\frac{1}{K}, 0)$. Using the Nyquist diagram shown in Figure 2 we need to check three cases, for $\frac{1}{K} < 0$ and $\frac{1}{K} > 0.05$, the point $(\frac{1}{K}, 0)$ is not encircled, i.e., $N = 0$. For $0 < \frac{1}{K} < 0.05$, the point $(\frac{1}{K}, 0)$ is encircled once in the clockwise direction, i.e., $N = 1$. In other words, the condition $N = -1$ is violated for any K and it follows that no K exists for which the closed loop system is stable.

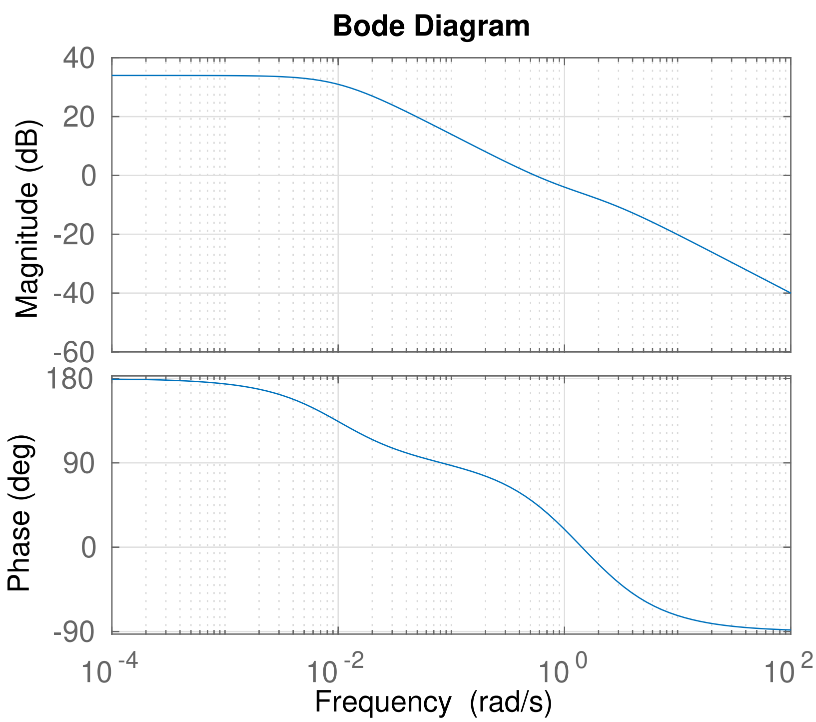

Figure 1: Bode diagram of the open-loop transfer function of the system Σ for $\alpha = -0.01$.

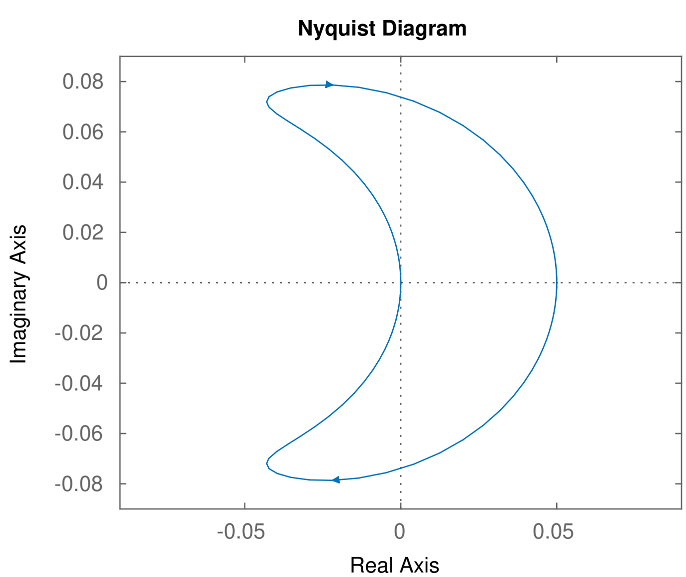

Figure 2: Nyquist diagram of the open-loop transfer function $G(s)$ of the system Σ for $\alpha = 10$. The dashed circles indicate magnitude in dB.