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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	Exercise
6	6	8	5	25 Points

1. We first apply Newton's law $F = m\ddot{x}$ on both masses to get

$$m_1\ddot{z}_1 = -k_1 z_1 - d_1\dot{z}_1 + d_2(\dot{z}_2 - \dot{z}_1)$$
$$m_2\ddot{z}_2 = -d_2(\dot{z}_2 - \dot{z}_1) + k_2(u - z_2)$$

Sorting terms and using the x representation, we get:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{m_1}(-k_1 x_1 - (d_1 + d_2)x_2 + d_2 x_4) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{m_2}(d_2 x_2 - k_2 x_3 - d_2 x_4 + k_2 u)
\end{aligned}$$

which leads finally to

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{d_1+d_2}{m_1} & 0 & \frac{d_2}{m_1} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{d_2}{m_2} & -\frac{k_2}{m_2} & -\frac{d_2}{m_2} \end{bmatrix} \\
B &= \begin{bmatrix} 0 & 0 & 0 & \frac{k_2}{m_2} \end{bmatrix}^T \\
C &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \\
D &= 0
\end{aligned}$$

2. First, we need $d_2 > 0$, otherwise the system decouples into two separate subsystems, of which one will not be observable. We can write down the observability matrix:

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{d_2}{m_2} & -\frac{k_2}{m_2} & -\frac{d_2}{m_2} \\ \frac{d_2}{m_2}(-\frac{k_1}{m_1}) & \frac{-d_2(m_2(d_1+d_2)+d_2 m_1)}{m_1 m_2^2} & \frac{k_2 d_2}{m_2^2} & \frac{d_2^2}{m_1 m_2} - \frac{k_2}{m_2} + \frac{d_2^2}{m_2^2} \end{bmatrix}$$

Due to the structure of the matrix, we can write the determinant as

$$\det(O) = \frac{k_1 d_2^2}{m_1 m_2^2}$$

Since the matrix is full rank whenever this is non-zero, this means that $k_1 > 0$ and $d_2 > 0$ are required for the system to be observable.

3. With $d_2 = 0$, the system dynamics matrix becomes

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{d_1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix} \tag{1}$$

and hence we have two independent systems, the first two states make one and the second two states make one. The eigenvalues of the two systems can be found using

$$\lambda\left(\lambda + \frac{d_1}{m_1}\right) + \frac{k_1}{m_1} = 0 \implies \lambda_{1,2} = \frac{-\frac{d_1}{m_1} \pm \sqrt{\left(\frac{d_1}{m_1}\right)^2 - 4\frac{k_1}{m_1}}}{2} \tag{2}$$

and

$$\lambda^2 + \frac{k_2}{m_2} = 0 \implies \lambda_{3,4} = \pm\sqrt{\frac{k_2}{m_2}}j \tag{3}$$

The input will be able to control states $x_3(t)$ and $x_4(t)$, but not the other two. Since the real part of $\lambda_{1,2}$ is smaller than 0, the uncontrollable modes are stable, which means the system is stabilizable. At the same time, we can measure $x_3(t)$. Since the real part of $\lambda_{3,4}$ is zero, these two states will just oscillate. Since we can measure them via $x_3(t)$, the system is also detectable.

4. We first write down the system in the new coordinates:

$$\begin{aligned} \dot{\hat{x}}(t) &= TAT^{-1}\hat{x}(t) + TBu(t), \\ y(t) &= CT^{-1}\hat{x}(t) + Du(t). \end{aligned}$$

Using the new matrices, we can derive that for $\hat{O}$ the observability matrix in the new coordinates and O the observability matrix in the old coordinates, it holds that

$$\hat{O} = \begin{bmatrix} CT^{-1} \\ CAT^{-1} \\ CA^2T^{-1} \\ CA^3T^{-1} \end{bmatrix} = OT^{-1}$$

which due to the invertability of T means that $\hat{O}$ has full rank iff O does.

Exercise 2

1	2	3	4	Exercise
5	8	4	8	25 Points

1. The transfer function can be computed from the state space representation as

$$\begin{aligned} G(s) &= C(s\mathbb{I} - A)^{-1}B + D \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-\sigma & -\omega_0 \\ \omega_0 & s-\sigma \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{(s-\sigma)^2 + \omega_0^2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-\sigma & \omega_0 \\ -\omega_0 & s-\sigma \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{\omega_0}{(s-\sigma)^2 + \omega_0^2}. \end{aligned}$$

The poles are $s_{1,2} = \sigma \pm j\omega_0$.

2. Instead of maximizing the magnitude of the transfer function directly, we can minimize the magnitude of the square of the denominator, as the numerator is constant.

$$\begin{aligned} \frac{d}{d\omega}|(j\omega - \sigma)^2 + \omega_0^2|^2 &= \frac{d}{d\omega}|(-\omega^2 + \omega_0^2 + \sigma^2) - 2j\omega\sigma|^2 \\ &= \frac{d}{d\omega}(-\omega^2 + \omega_0^2 + \sigma^2)^2 + 4\omega^2\sigma^2 \\ &= -4\omega(-\omega^2 + \omega_0^2 + \sigma^2) + 8\omega\sigma^2 \overset{!}{=} 0 \\ \Rightarrow\ & \omega \cdot (\omega^2 - (\omega_0^2 - \sigma^2)) = 0 \\ \Rightarrow\ & \omega_1 = 0 \ \vee\ \omega_2 = \sqrt{\omega_0^2 - \sigma^2} \end{aligned}$$

We thus have two candidates (ω_1 and ω_2) for maximizing the magnitude (i.e., minimizing the denominator). To find the maximum we evaluate the denominator $D(\omega) := (-\omega^2 + \omega_0^2 + \sigma^2)^2 + 4\omega^2\sigma^2$ for both candidates and see which one is smaller.

- $D(\omega_1) = D(0) = (\omega_0^2 + \sigma^2)^2$
- $D(\omega_2) = D(\sqrt{\omega_0^2 - \sigma^2}) = 4\sigma^2(\omega_0^2 - \sigma^2)$

Since $\omega_0 > \sigma$ we have $D(0) > 4\sigma^2 \geq D(\sqrt{\omega_0^2 - \sigma^2})$. Hence, $\omega_2 = \sqrt{\omega_0^2 - \sigma^2}$ is maximizing the magnitude.

3.

$$
\begin{aligned}
|G(j\omega_c)| &= \left|\frac{\omega_0}{(j\sqrt{\omega_0^2-\sigma^2}-\sigma)^2+\omega_0^2}\right| \\
&= \frac{\omega_0}{|-\omega_0^2+\sigma^2-2j\sqrt{\omega_0^2-\sigma^2}\sigma+\sigma^2+\omega_0^2|} \\
&= \frac{\omega_0}{|2\sigma^2-2j\sqrt{\omega_0^2-\sigma^2}\sigma|} \\
&= \frac{\omega_0}{\sqrt{4\sigma^4+4\omega_0^2\sigma^2-4\sigma^4}} \\
&= \frac{\omega_0}{2\omega_0\sigma} = \frac{1}{2\sigma}
\end{aligned}
$$

4. Based on ω_c and $|G(j\omega_c)|$ we find the following pairings as the solution: 1c(i), 2a(iii), 3b(ii).

Exercise 3

1	2	3	4	5	Exercise
2	4	6	6	7	25 Points

1. The system is linear (linear ODE) but time-varying, since there is explicit dependence on time, see (e^{-t})-term.

2. The state space form is given by

$$\frac{d}{dt}x(t) = \begin{pmatrix} x_2(t) \\ -\theta x_2(t) - x_1(t)(1 + x_3^2(t)) \\ -\frac{1}{2}x_3(t) \end{pmatrix},$$

   which is nonlinear, see for example the x_3^2-term. We need the initial condition $x_3(0) = 1$.

3. There is only one equilibrium which is $\hat{x} = (0, 0, 0)^T$. We compute the Jacobi matrix

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -x_3^2 - 1 & -\theta & -2x_1x_3 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix},$$

   that evaluated at the origin is

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & -\theta & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix},$$

   whose eigenvalues are $\{-\frac{1}{2}, \frac{-\theta+\sqrt{\theta^2-4}}{2}, \frac{-\theta-\sqrt{\theta^2-4}}{2}\}$. Hence, the origin is locally asymptotically stable if $\theta > 0$ and it is unstable if $\theta < 0$.

4.

$$\frac{d}{dt}V(x(t)) = \frac{2x_2^2}{x_3^2 + 1}\left(-\theta + \frac{x_3^2}{x_3^2 + 1}\right) - \alpha x_3^2$$

   Hence for all $x \in S := \left\{x \in \mathbb{R}^3 \ : \ \frac{x_3^2}{x_3^2+1} \leq \theta\right\}$ the Lie-deriative is less than or equal to zero.

5. When considering the open set $\text{int}(S) \neq \{\}$, since $\theta > 0$, by using Theorem 7.2 from the lecture notes we get that the origin is stable, as

   - $V(0) = 0$
   - $V(x) > 0$ for all $x \in \text{int}(S)$ and $x \neq 0$
   - $\frac{d}{dt}V(x(t)) \leq 0$ for all $x \in \text{int}(S)$ by the previous subtask.

Exercise 4

1	2	3	4	5	Exercise
3	6	4	6	6	25 Points

1. The poles of the open loop system are the eigenvalues of the system matrix, which can immediately be determined as $\lambda_1 = 2$ and $\lambda_2 = 1$, because the system matrix is triangular. Since the magnitude of one eigenvalue is larger than 1, the system is open loop unstable.

2. The closed loop system is given as

$$x[k+1] = Ax[k] - BKx[k] = (A - BK)x[k]$$

For the given system and controller, the poles can be computed as

$$\begin{aligned} \det(\lambda \mathbf{I} - (A - BK)) =& \det\left(\begin{bmatrix} \lambda - 2 & -1 \\ \frac{9}{4} & \lambda + 1 \end{bmatrix}\right) \stackrel{!}{=} 0, \\ \Leftrightarrow \quad & \lambda^2 - \lambda + \frac{1}{4} = 0 \end{aligned}$$

which implies $p_1 = p_2 = \frac{1}{2}$. Therefore, the closed loop system is asymptotically stable.

3. To check observability, we determine the observer matrices $C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}$ which correspond to the output choices and compute the observability matrices

$$\mathcal{O}_1 = \begin{bmatrix} C_2 \\ C_2 A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

and

$$\mathcal{O}_2 = \begin{bmatrix} C_1 \\ C_1 A \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

It can be seen that the pair (C_2, A) is not observable since $\text{rank}(O_2) = 1 < 2$, but the pair (C_1, A) is observable, since $\text{rank}(O_1) = 2$. Therefore, the output that only has $x_1[k]$ has to be chosen.

4. The estimation error is given as

$$e[k+1] = x[k+1] - \hat{x}[k+1] = Ae[k] - LCe[k] = (A - LC)e[k]$$

5. For the given system, the poles of the estimation error dynamics can be computed as

$$\begin{aligned} \det(\lambda \mathbf{I} - (A - LC)) = \det & \left(\begin{bmatrix} \lambda - 2 + l_1 & -1 \\ l_2 & \lambda - 1 \end{bmatrix}\right) \stackrel{!}{=} \left(\lambda - \frac{1}{4}\right)^2 \\ \Leftrightarrow \quad & \lambda^2 + (-3 + l_1)\lambda + (2 - l_1 + l_2) = \lambda^2 - \frac{1}{2}\lambda + \frac{1}{16} \\ \Rightarrow \quad & l_1 = \frac{5}{2}, \quad l_2 = \frac{9}{16} \end{aligned}$$