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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	Exercise
8	6	5	4	2	25 Points

1. We first write down the equations for voltage and current for the capacitors and the inductor

$$i_{C_1}(t) = C_1 \frac{d}{dt} v_{C_1}(t), \qquad i_{C_2}(t) = C_2 \frac{d}{dt} v_{C_2}(t), \qquad v_L(t) = L \frac{d}{dt} i_L(t). \tag{1}$$

With these equations and the relationships

$$i_R(t) = \frac{v_{C_2}(t)}{R}, \qquad i_L(t) = i_{C_2}(t) + i_R(t), \qquad V_{\text{in}}(t) = v_{C_1}(t) + v_{C_2}(t) + v_L(t)$$

we can write down equations for the derivatives of v_{C_1}, v_{C_2} and i_L:

$$\frac{d}{dt} i_L(t) = \frac{1}{L} v_L(t) = \frac{1}{L}\Big[V_{\text{in}}(t) - v_{C_1}(t) - v_{C_2}(t)\Big], \tag{2a}$$

$$\frac{d}{dt} v_{C_1}(t) = \frac{1}{C_1} i_{C_1}(t) = \frac{1}{C_1} i_L(t), \tag{2b}$$

$$\frac{d}{dt} v_{C_2}(t) = \frac{1}{C_2} i_{C_2}(t) = \frac{1}{C_2}\left[i_L(t) - \frac{v_{C_2}(t)}{R}\right], \tag{2c}$$

which leads to the state-space equations

$$\frac{d}{dt} x(t) = \begin{bmatrix} 0 & -\frac{1}{L} & -\frac{1}{L} \\ \frac{1}{C_1} & 0 & 0 \\ \frac{1}{C_2} & 0 & -\frac{1}{RC_2} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} u(t) \tag{3a}$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(t) \tag{3b}$$

2. We first note that the characteristic polynomial for 3 eigenvalues at -1 is written as follows:

$$\begin{aligned}(\lambda+1)^3 &= (\lambda+1)(\lambda^2+2\lambda+1) \\ &= \lambda^3 + 2\lambda^2 + \lambda + \lambda^2 + 2\lambda + 1 \\ &= \lambda^3 + 3\lambda^2 + 3\lambda + 1\end{aligned} \tag{4}$$

We can then compute

$$\begin{aligned}&\det\Big(\lambda I - (A - KC)\Big) = \det \begin{bmatrix} \lambda & 1 & 1+k_1 \\ -1 & \lambda & k_2 \\ -1 & 0 & \lambda+1+k_3 \end{bmatrix} \\ &= \lambda\Big(\lambda(\lambda+1+k_3)\Big) - \Big((-1)(\lambda+1+k_3) + k_2\Big) + (1+k_1)(\lambda) \\ &= \lambda^3 + (1+k_3)\lambda^2 + \lambda + 1 + k_3 - k_2 + \lambda + k_1\lambda \\ &= \lambda^3 + (1+k_3)\lambda^2 + (2+k_1)\lambda + (1+k_3-k_2)\end{aligned} \tag{5}$$

hence comparing (4) and (5), we get

$$k_1 = 1, \qquad k_2 = 2, \qquad k_3 = 2. \tag{6}$$

3. We compute the observability and controllability matrices:

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \tag{7}$$

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

While P has full rank, Q is rank-deficient. This means the system is controllable, but not observable.

4. To determine whether the system is detectable, we need to look at the rank of

$$M_i = \begin{bmatrix} C \\ \lambda_i I - A \end{bmatrix}$$

for each eigenvalue λ_i of A. If $\text{rank}(M_i) = 3$ for all non-negative eigenvalues of A, the system is detectable. We have $\lambda_i = \pm i$ here, so we check both cases. For $\lambda = \pm i$, we have

$$\text{rank} \begin{bmatrix} C \\ \lambda I - A \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \pm i & 1 & 1 \\ -1 & \pm i & 0 \\ 0 & 0 & \pm i + 1 \end{bmatrix} = 2 \tag{9}$$

which means the system is not detectable.

Exercise 2

1	2	3	4	5	Exercise
6	5	2	5	7	25 Points

1. The eigenvalues are the roots of the characteristic polynomial $\lambda I - A$, whose determinant is given by

$$\det|\lambda I - A| = \det\left|\begin{bmatrix}\lambda-\sigma & -\omega \\ \omega & \lambda-\sigma\end{bmatrix}\right| \tag{10}$$
$$= (\lambda-\sigma)^2 + \omega^2 \tag{11}$$

This gives us $\lambda_1 = \sigma + j\omega$ and $\lambda_2 = \sigma - j\omega$.

Let $V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$, where v_1 and v_2 are the eigenvectors. Then, these can be computed by solving $\lambda_i v_i = Av_i$ for $i = 1, 2$. Thus, we obtain

$$V = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$$

2. If Φ is the state transition matrix, it suffices to show that $\frac{d\Phi}{dt} = A\Phi$. Note that

$$\begin{aligned}\frac{d\Phi}{dt} &= \sigma\Phi + \exp(\sigma t)\begin{bmatrix}-\omega\sin(\omega t) & \omega\cos(\omega t) \\ -\omega\cos(\omega t) & -\omega\sin(\omega t)\end{bmatrix} \\ &= \exp(\sigma t)\begin{bmatrix}\sigma\cos(\omega t) - \omega\sin(\omega t) & \sigma\sin(\omega t) + \omega\cos(\omega t) \\ -\sigma\sin(\omega t) - \omega\cos(\omega t) & \sigma\cos(\omega t) - \omega\sin(\omega t)\end{bmatrix}\end{aligned}$$

Also note that

$$\begin{aligned}A\Phi &= \exp(\sigma t)\begin{bmatrix}\sigma & \omega \\ -\omega & \sigma\end{bmatrix}\begin{bmatrix}\cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t)\end{bmatrix} \\ &= \exp(\sigma t)\begin{bmatrix}\sigma\cos(\omega t) - \omega\sin(\omega t) & \sigma\sin(\omega t) + \omega\cos(\omega t) \\ -\sigma\sin(\omega t) - \omega\cos(\omega t) & \sigma\cos(\omega t) - \omega\sin(\omega t)\end{bmatrix}\end{aligned}$$

Thus, we conclude that the state transition matrix is indeed given by Φ.

3. The system is asymptotically stable when $\sigma < 0$ and stable when $\sigma \leq 0$.

4. The controllability matrix for this system is given by $\begin{bmatrix} B & AB \end{bmatrix}$. Computing its determinant, we get

$$\det\left|\begin{bmatrix} b_1 & \sigma b_1 + \omega b_2 \\ b_2 & -\omega b_1 + \sigma b_2\end{bmatrix}\right| = -\omega(b_1^2 + b_2^2)$$

For $\omega = 0$, the system is uncontrollable for any B. For $\omega \neq 0$, the system is controllable as long as $B \neq 0$.

5. When $\omega \neq 0$, the system has been shown to be controllable. Thus, all complex conjugate pairs of eigenvalues can be obtained.

   When $\omega = 0$, the characteristic polynomial corresponding to $A_{CL} = A - BK$ is given by

$$\begin{aligned}\det|\lambda_{CL}I - A_{CL}| &= \det\left|\begin{bmatrix}\lambda_{CL}-\sigma & -\omega \\ \omega - k_1 & \lambda_{CL}-\sigma-k_2\end{bmatrix}\right| \\ &= (\lambda_{CL}-\sigma)^2 - k_2(\lambda_{CL}-\sigma)\end{aligned}$$

   where we set $\omega = 0$ to obtain the last equality. Solving for the roots, we get $\lambda_{CL} = \{\sigma + k_2, \sigma\}$. Thus, one of the closed-loop eigenvalues remains at σ for any K, which explains why the system is uncontrollable when $\omega = 0$. The second eigenvalue can be placed anywhere by appropriately choosing k_2. The value of k_1 is immaterial.

Exercise 3

1	2	3	4	5	Exercise
3	2	5	7	8	25 Points

1. The system is linear if g and h are linear functions, it is nonlinear otherwise **(1 point)**. It is always autonomous **(1 point)** and time invariant **(1 point)**.

2. The system is at an equilibrium point when $\dot{x} = \dot{y} = 0$ **(0.5 point)**. Note that **(1 point)**
$$|g(0)| \leq 0 \text{ and } |h(0)| \leq 0 \Rightarrow g(0) = h(0) = 0.$$
Consequently, if $x = y = 0$ then **(0.5 point)**
$$\dot{x} = -0 + g(0) = 0 \text{ and } \dot{y} = -0 + h(0) = 0$$
implying that $(0, 0)$ is an equilibrium point.

3. (a) By problem definition g and h are continuously differentiable therefore the limit in zero exists **(0.5 point)** and can be computed as **(0.5 point)**
$$h'(0) = \lim_{\varepsilon \to 0^+} \frac{h(\varepsilon) - h(0)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{h(\varepsilon)}{\varepsilon}$$
Note that
$$|h(\varepsilon)| \leq \frac{|\varepsilon|}{2} = \frac{\varepsilon}{2} \Rightarrow -\frac{\varepsilon}{2} \leq h(\varepsilon) \leq \frac{\varepsilon}{2} \Rightarrow -\frac{1}{2} \leq \frac{h(\varepsilon)}{\varepsilon} \leq \frac{1}{2} \textbf{(1 point)}$$
Since the relation above holds for all $\varepsilon > 0$ we have $-\frac{1}{2} \leq \lim_{\varepsilon \to 0^+} \frac{h(\varepsilon)}{\varepsilon} \leq \frac{1}{2}$ **(1 point)**, hence $|h'(0)| \leq \frac{1}{2}$. Same holds for g.

   (b) From $(|x| - |y|)^2 \geq 0$ we get
$$x^2 + y^2 - 2|xy| \geq 0 \Rightarrow x^2 + y^2 \geq 2|xy|.$$
(2 point)

4. The matrix defining the linearized system is
$$A(x, y) := \begin{bmatrix} -1 & g'(y) \\ h'(x) & -1 \end{bmatrix}.$$
Computing $A(0, 0)$ yields
$$A(0, 0) := \begin{bmatrix} -1 & g'(0) \\ h'(0) & -1 \end{bmatrix}, \qquad \textbf{(2 points: 0.5 each)}$$
the eigenvalues are the roots of the characteristic polynomial
$$(s + 1)^2 - g'(0)h'(0) = s^2 + 2s + 1 - g'(0)h'(0),$$
which are
$$\lambda_{1,2} = -1 \pm \sqrt{1 - 1 + g'(0)h'(0)} = -1 \pm \sqrt{g'(0)h'(0)} \textbf{(2 points: 1 each)}$$
We distinguish two cases:

- if $g'(0)h'(0) < 0$ the eigenvalues are imaginary with real part -1. **(1 point)**
- If $g'(0)h'(0) \geq 0$ then $0 \leq g'(0)h'(0) = |g'(0)||h'(0)| \leq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Hence $Re(\lambda_{1,2}) \leq -\frac{3}{4} < 0$. **(1 point)**

Hence in both case the real part of the eigenvalues is negative **(0.5 point)** and the equilibrium point at the origin is locally asymptotically stable **(0.5 point)**.

5. Consider the function $V(x,y) = \frac{x^2+y^2}{2}$.

- $V(x,y) > 0$ for all $(x,y) \neq (0,0)$ **(0.5 point)** and $V(0,0) = 0$ **(0.5 point)** .
- The Lie derivative of $V(x,y)$ is

$$\begin{aligned}
\dot{V}(x,y) &= \frac{\partial V(x,y)}{\partial x}\dot{x} + \frac{\partial V(x,y)}{\partial y}\dot{y} \ \textbf{(0.5 point)} \\
&= x(-x+g(y)) + y(-y+h(x)) = -x^2 - y^2 + xg(y) + yh(x) \textbf{(0.5 point)} \\
&\leq -x^2 - y^2 + |x||g(y)| + |y||h(x)| \leq -x^2 - y^2 + |x|\frac{|y|}{2} + |y|\frac{|x|}{2} \\
&\leq -x^2 - y^2 + |xy| \textbf{(1 point)} \leq -x^2 - y^2 + \frac{x^2+y^2}{2} \\
&= -\frac{x^2+y^2}{2} \textbf{(1 point)} < 0, \quad \forall (x,y) \neq (0,0) \textbf{(1 point)}
\end{aligned}$$

Hence by Theorem 7.2 **(1 point)** the equilibrium point $(0,0)$ is globally **(1 point)** asymptotically stable **(1 point)**.

Exercise 4

1	2	3	4	5	6	Exercise
4	4	8	3	3	3	25 Points

1. The system matrix of the feedforward stage is triangular, which means that the eigenvalues are the entries on the diagonal $\lambda_1 = \lambda_2 = 0 < 1$ **(1 point)**. Therefore, the feedforward stage is asymptotically stable for all possible parameter choices **(1 point)**. To check observability, we compute the observability matrix:

$$\mathcal{O} = \begin{pmatrix} C_1 \\ C_1 A_1 \end{pmatrix} = \begin{pmatrix} b_2 & b_1 \\ 0 & b_2 \end{pmatrix} \quad \textbf{(1 point)}$$

   It is easy to see that this matrix has full rank if and only if $b_2 \neq 0$ **(1 point)**.

2. First, compute the eigenvalues of the system matrix:

$$\det(\lambda I - A) = \lambda^2 - a_2 \stackrel{!}{=} 0 \;\Rightarrow\; \lambda_{1,2} = \pm\sqrt{a_2} \quad \textbf{(1 point)}$$

   Therefore, the system is asymptotically stable if and only if $|a_2| < 1$ **(1 point)**. To check observability, we compute the observability matrix:

$$\mathcal{O} = \begin{pmatrix} C_2 \\ C_2 A_2 \end{pmatrix} = \begin{pmatrix} 0 & a_2 \\ a_2 & 0 \end{pmatrix} \quad \textbf{(1 point)}$$

   It is easy to see that this matrix has full rank if and only if $a_2 \neq 0$ **(1 point)**.

3. Using the state $\xi(k) := \begin{pmatrix} x(k)^\top & z(k)^\top \end{pmatrix}^\top$, the system matrices of the complete system are given as:

$$\xi(k+1) = \begin{pmatrix} A_2 & B_2 C_1 \\ 0 & A_1 \end{pmatrix} \xi(k) + \begin{pmatrix} B_2 D_1 \\ B_1 \end{pmatrix} u(k)$$
$$y(k) = \begin{pmatrix} C_2 & D_2 C_1 \end{pmatrix} \xi(k) + D_1 D_2 u(k) \quad \textbf{(1 point per matrix)}$$

   and filling in the parameters, we obtain:

$$\xi(k+1) = \begin{pmatrix} 0 & a_2 & b_2 & b_1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xi(k) + \begin{pmatrix} b_0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u(k)$$
$$y(k) = \begin{pmatrix} 0 & a_2 & b_2 & b_1 \end{pmatrix} \xi(k) + \begin{pmatrix} b_0 \end{pmatrix} u(k)$$

   (1 point per matrix)

   If another approach is chosen, the 4 points for terms in the first equation can be granted for similar intermediate results.

4. The complete system is given as a concatenation of two systems **(1 point)** and therefore, it is asymptotically stable if and only if the two subsystems are asymptotically stable **(1 point)**. Here, this means that the system is asymptotically stable if

and only if $|a_2| < 1$ **(1 point)** (using the optimality condition for the feedback stage derived earlier). Alternatively, one can compute $\det(\lambda I - A)$ directly, which is easy if the appropriate row is expanded. In this case, up to two points can be awarded for intermediate results and 1 point for the correct end result.

5. The z-transform of the difference equations is given as:

$$Y(z) - a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) = b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z)$$
$$\Leftrightarrow G(z) := \frac{Y(z)}{U(z)} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

   Both the numerator and the denominator of $G(z)$ are of order two, thus the original difference equation describes a second order system. It follows that a minimal state space realization has two states. **(1 point for correct answer, 2 points for justification)**

6. The state space realization uses four states to implement a second order system **(1 point)**. Therefore, there must be either uncontrollable or unobservable modes **(1 point)**. If the system is controllable, it follows immediately that it is not observable **(1 point)**.