Automatic Control Laboratory
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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	Exercise
8	4	2	7	4	25 Points

1. We first write down the equations for voltage and current for the capacitor and inductor
$$v_C(t) = \frac{1}{C_1}\int_0^t i_C(\tau)d\tau, \qquad v_L(t) = L_1\frac{d}{dt}i_L(t). \tag{1}$$
With these equations and the relationships
$$i_C(t) = i_R(t) + i_L(t), \qquad v_L(t) = V_{\text{in}}(t) - v_C(t),$$
we can write down equations for the derivatives of v_C and i_L
$$\frac{d}{dt}v_C(t) = \frac{1}{C_1}i_C(t) = \frac{1}{C_1}\Big[i_L(t) + i_R(t)\Big] \tag{2a}$$
$$\frac{d}{dt}i_L(t) = \frac{1}{L_1}v_L(t) = \frac{1}{L_1}\Big[V_{\text{in}}(t) - v_C(t)\Big]. \tag{2b}$$
Next, we write the current i_R as a function of the other variables:
$$i_R = \frac{V_{\text{in}} - v_C}{R_1 + R_2} \tag{3}$$
and notice the output voltage can be written as
$$V_{\text{out}} = i_R R_2, \tag{4}$$
which finally leads to the state-space equations
$$\dot{x} = \begin{bmatrix} \frac{-1}{C_1(R_1+R_2)} & \frac{1}{C_1} \\ -\frac{1}{L_1} & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{C_1(R_1+R_2)} \\ \frac{1}{L_1} \end{bmatrix} u \tag{5a}$$
$$y = \begin{bmatrix} \frac{-R_2}{R_1+R_2} & 0 \end{bmatrix} x + \frac{R_2}{R_1+R_2}u \tag{5b}$$

2. Notice first that with the given values, A, B, C, D become the ones specified in Task 2. We can simply compute the controllability matrix P and the observability matrix Q:
$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{15}{16} \\ 1 & -\frac{1}{4} \end{bmatrix}, \qquad Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{8} & -\frac{1}{2} \end{bmatrix} \tag{6}$$
A simple determinant computation reveals that since $\det(P) = -1 \neq 0$ and $\det(Q) = \frac{1}{4} \neq 0$, P and Q have full rank, and the system is both controllable and observable.

3. Intuitive argument: Since the output resistor is 0Ω, the output voltage will always be 0, hence we cannot observe what is happening in the circuit. Mathematically, this simply makes $C = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and hence Q all zeros with rank 0.
However, the system is still detectable, since none of its states are unstable. The intuitive argument for this is that it is physically a dissipative system: No external energy sources are present and R_1 will dissipate any energy in the system over time.

4. The transfer function can be calculated using the formula

$$
\begin{aligned}
G(s) &= C(sI - A)^{-1}B + D \\
&= \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\frac{1}{4} & 1 \\ -1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} + \frac{1}{2} \\
&= \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix} \left(\begin{bmatrix} s + \frac{1}{4} & -1 \\ 1 & s \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} + \frac{1}{2} \\
&= \frac{1}{s^2 + \frac{1}{4}s + 1} \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ -1 & s + \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} + \frac{1}{2} \\
&= \frac{1}{s^2 + \frac{1}{4}s + 1} \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4}s + 1 \\ s \end{bmatrix} + \frac{1}{2} \\
&= \frac{-\frac{1}{8}s - \frac{1}{2}}{s^2 + \frac{1}{4}s + 1} + \frac{1}{2} \\
&= \frac{2s^2}{4s^2 + s + 4}
\end{aligned} \tag{7}
$$

5. The simple answer for this task is that in steady state, there can be no (DC) current over the capacitor C_1 and hence also none over L_1 or the resistors. Hence for any constant input voltage $V_{\text{in}} = V_0$, the steady-state output voltage V_{out} will be 0. Mathematically, this can be shown by looking at the DC gain of the transfer function:

$$G(0) = \frac{0}{4} = 0$$

meaning the DC gain of the system is 0.

Exercise 2

1	2	3	4	5	Exercise
3	5	8	5	4	25 Points

1. The system is of second order (degree of denominator polynomial). From the Hurwitz-criterion for second-order transfer functions it follows that $\omega > 0$ and $\zeta > 0$ are necessary and sufficient for asymptotic stability.

   Note that we specified $\omega \geq 0$ in the beginning of the exercise to simplify an interpretation of ω as a system frequency. If we allowed $\omega < 0$, the condition would be that ω and ξ have the same sign and neither is equal to zero.

2. Since
$$G(s) = \frac{Y(s)}{U(s)}$$
   it follows that
$$(s^2 + 2\omega\zeta s + \omega^2)Y(s) = \omega^2 U(s)$$
   which corresponds to the second order ODE
$$\ddot{y}(t) + 2\omega\zeta\dot{y}(t) + \omega^2 y(t) = \omega^2 u(t).$$

3. For the given states we obtain the state space realization
$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\omega\zeta \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0.$$
   By coordinate transformations $\hat{x}(t) = Tx(t)$ with $\det(T) \neq 0$ we can obtain infinitely many equivalent state space realizations of the transfer function (4) with two states.

   By introducing unobservable or uncontrollable states, e.g. a third state $x_3(t)$ that does not affect $x_1(t)$, $x_2(t)$, and $y(t)$, we could construct higher-order realizations of (4).

4. The desired statespace description of the closed-loop system is
$$\dot{x}(t) = Ax(t) + Bu(t) = (A + BK)x(t) = \begin{bmatrix} 0 & 1 \\ K_1 - \omega^2 & K_2 - 2\omega\zeta \end{bmatrix}$$
$$y(t) = Cx(t).$$
   The entries of the C matrix and the first row of $A + BK$ already match the desired values. From the second row of $A + BK$ we obtain
$$K_1 = 0, \qquad K_2 \leq 2\omega\zeta - 5.$$

5. Note that the given realization (5) of the system is in controllable canonical form and hence controllable. Therefore, the poles can be placed arbitrarily with state feedback. In particular, they can be placed at -1 and -2.

Exercise 3

1	2	3	4	Exercise
3	4	8	10	25 Points

1. The system is non linear **(1 point)**, autonomous **(1 point)** and time invariant **(1 point)**.

2. The system is at an equilibrium point when $\dot{x} = \dot{y} = 0$. These conditions imply

$$\begin{aligned} \dot{x} = 0 \quad &\Rightarrow \quad x = 0 \quad \text{or} \quad y = 1 \qquad \textbf{(1 point)} \\ \dot{y} = 0 \quad &\Rightarrow \quad y = 0 \quad \text{or} \quad x = 1 \qquad \textbf{(1 point)}. \end{aligned}$$

   Hence the only equilibrium points are the origin **(1 point)** and $(\bar{x}, \bar{y}) := (1, 1)$ **(1 point)**.

3. The matrix defining the linearized system is

$$A(x, y) := \begin{bmatrix} 1 - y & -x \\ y & x - 1 \end{bmatrix}. \qquad \textbf{(2 points: 0.5 each)}$$

   Computing $A(0, 0)$ yields

$$A(0, 0) := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \textbf{(1 point)}$$

   the eigenvalues are $1 > 0$ and $-1 < 0$ **(1 point: 0.5 each)**. Hence the equilibrium point at the origin is unstable. **(1 point)**.
   Computing $A(\bar{x}, \bar{y})$ yields

$$A(\bar{x}, \bar{y}) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \textbf{(1 point)}$$

   the eigenvalues are $\pm i$ **(1 point: 0.5 each)**. Since they have zero real part it is not possible to conclude anything about the stability of $(\bar{x}, \bar{y})$ form the linearization technique. **(1 point)**

4. Consider the function $V(x, y) = -xye^{-(x+y)} + e^{-2}$.

   (a) $V(0, 0) = 0 + e^{-2} > 0$ **(1 point)**
   $V(\bar{x}, \bar{y}) = 0$. **(1 point)**

   (b) We start by computing the first order derivative of $V(x, y)$ **(2 points: 1 each)**.

$$\frac{\partial V(x, y)}{\partial x} = -ye^{-(x+y)}(1 - x)$$

$$\frac{\partial V(x, y)}{\partial y} = -xe^{-(x+y)}(1 - y)$$

The Lie derivative of $V(x,y)$ is

$$\begin{aligned}\dot{V}(x,y) &= \frac{\partial V(x,y)}{\partial x}\dot{x} + \frac{\partial V(x,y)}{\partial y}\dot{y} \\ &= \left[-ye^{-(x+y)}(1-x)\right][x-xy] + \left[-xe^{-(x+y)}(1-y)\right][xy-y] \\ &= \left[-xye^{-(x+y)}(1-x)(1-y)\right] + \left[-xye^{-(x+y)}(1-y)(x-1)\right] = 0.\end{aligned}$$

Hence by Theorem 7.2 the equilibrium point $(\bar{x}, \bar{y})$ is locally stable. **(3 points)**.

(c) Since the Lie derivative of $V(x,y)$ is always equal to zero, the level sets $V(x,y) = c$, for any constant $c \in \mathbb{R}_+$, are invariant sets. **(1 point)**. By noticing that $\dot{x} > 0$ iff $y < 1 = \bar{y}$ and $\dot{y} > 0$ iff $x > 1 = \bar{x}$ **(1 point)**, it can be concluded that the trajectories circulate onto the level set counter clock wise **(1 point)**, see Figure 1.

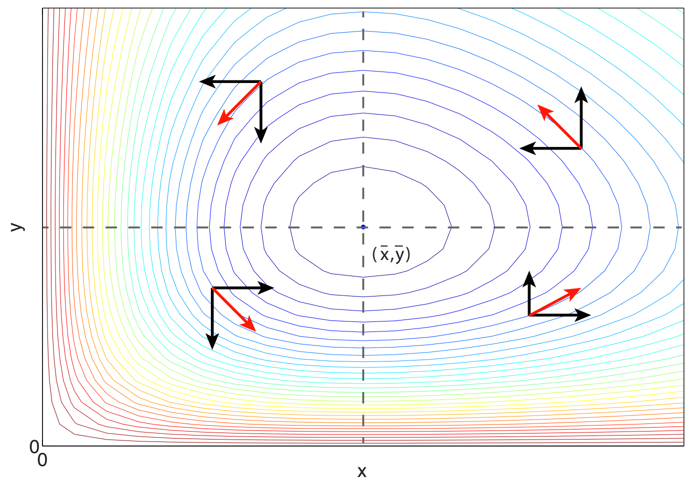

Figure 1: Trajectories of the system.

Exercise 4

1	2	3	4	Exercise
3	11	8	3	25 Points

1. To verify controllability, we compute the controllability matrix:

$$\mathcal{P} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & a_3 \\ 1 & a_3 & a_2 + a_3^2 \end{bmatrix}$$

   We see that $\det(\mathcal{P}) = -1 \neq 0$, thus $\text{rank}(\mathcal{P}) = 3$ and the system is controllable for all parameter values a_1, a_2 and a_3. Alternatively, one can also recognize that the system is in controllable–canonical form and thus controllable for all parameter values a_1, a_2 and a_3. **(1 point for correct answer + 2 points for justification)**

2. (a) By plugging in the equation for state–feedback $u(k) = Kx(k)$ in the systems equation **(1 point)**, we find

$$x(k+1) = Ax(k) + BKx(k) = (A + BK)\, x(k)$$

   so

$$A_K = A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 + k_1 & a_2 + k_2 & a_3 + k_3 \end{bmatrix} \textbf{(1 point)}.$$

   (b) The characteristic polynomial of the closed–loop is defined as $\pi(\lambda) = \det(\lambda I_3 - A_K)$ **(1 point)** and it can be computed as:

$$\begin{aligned} \det(\lambda I_3 - A_K) &= \det\left(\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -a_1 - k_1 & -a_2 - k_2 & \lambda - a_3 - k_3 \end{bmatrix}\right) \\ &= \lambda^2(\lambda - a_3 - k_3) - a_1 - k_1 - \lambda(a_2 + k_2) \\ &= \lambda^3 + (-a_3 - k_3)\lambda^2 + (-a_2 - k_2)\lambda + (-a_1 - k_1). \quad \textbf{(2 points)} \end{aligned}$$

   As an alternative to evaluating the determinant, one can also find the characteristic polynomial of a system in controllable canonical form in the slides.

   (c) The desired characteristic polynomial with poles at λ_1, λ_2 and λ_3 is given as:

$$\begin{aligned} \pi(\lambda) &\overset{!}{=} (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) && \textbf{(1 point)} \\ &= \lambda^3 + (-\lambda_1 - \lambda_2 - \lambda_3)\lambda^2 \dots \\ &\quad \dots + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda + (-\lambda_1\lambda_2\lambda_3), && \textbf{(1 point)} \end{aligned}$$

   By comparing it to the characteristic polynomial of the closed loop, we find:

$$\begin{aligned} k_1 &= -a_1 + \lambda_1\lambda_2\lambda_3, \\ k_2 &= -a_2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3, \\ k_3 &= -a_3 + \lambda_1 + \lambda_2 + \lambda_3. && \textbf{(1 point each)} \end{aligned}$$

From the previous computations, it is evident that we can find parameters k_1, k_2 and k_3 for any choice of $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, the closed–loop eigenvalues can be arbitrarily placed **(1 point)**.

3. The evolution of the autonomous closed–loop system is given as $x(k) = A_K^k x_0$ **(2 points)**. We can achieve deadbeat behavior of the closed–loop $x(n) = A_K^n x_0 = 0$ by making the closed–loop system matrix A_K nilpotent **(2 points)**. According to the hint, this can be achieved by placing all closed–loop poles at zero: $\lambda_1 = \lambda_2 = \lambda_3 = 0$ **(2 points)**.

   We can use the result to Question 2 for pole placement and find that $k_1 = -a_1$, $k_2 = -a_2$ and $k_3 = -a_3$ **(2 points)** yields a deadbeat controller for the given system. This controller can also be found by inspection of the closed–loop system matrix, it is easy to see that it becomes nilpotent for these controller parameters.

4. No. The closed–loop system consisting of a linear system and a linear controller is again linear **(1.5 points)**. Continuous–time linear systems *never* exhibit deadbeat behavior **(1.5 points)**.