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D-ITET
Examination Winter 2013/2014
08.02.2014

Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	Exercise
2	4	4	5	10	25 Points

1. **[2 Points]** The system can be rewritten as

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{1}$$

where

$$A = \begin{pmatrix} 2\beta & \beta^2 \\ 1 & 2\beta \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Therefore, the system clearly is linear. Since the matrices A, B, C do not depend on time, the system is time invariant.

2. **[4 Points]** The observability matrix is given by $Q = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2\beta+1 & \beta^2+2\beta \end{pmatrix}$. The system is observable if and only if Q has rank two, which is equivalent to $\det(Q) \neq 0$.

$$\det(Q) = \beta^2 + 2\beta - (2\beta + 1) = \beta^2 - 1,$$

and therefore the system is observable for all $\beta \in \mathbb{R} \setminus \{1, -1\}$.

3. **[4 Points]** The controllability matrix is given by $P = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 2\beta \\ 0 & 1 \end{pmatrix}$. The system is controllable if and only if P has rank two, which is equivalent to $\det(R) \neq 0$.

$$\det(R) = 1,$$

hence the system is controllable for any $\beta \in \mathbb{R}$.

4. **[5 Points]** We compute the Eigenvalues of the system matrix A

$$\det(A - \lambda I_2) = \lambda^2 - 4\beta\lambda + 3\beta^2 = 0$$

$\Rightarrow \lambda_1 = 3\beta$ and $\lambda_2 = \beta$. Since the two eigenvalues are distinct the matrix A is diagonalizable and as such we know from the lecture notes that the system is asymptotically stable if and only if $\beta < 0$.

5. (a) **[5 Points]**

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ &= Ax(t) + B(Kx(t) + v(t)) \\ &= (A + BK)x(t) + Bv(t) \\ &= \begin{pmatrix} 4+k_1 & 4+k_2 \\ 1 & 4 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v(t) \\ y(t) &= \begin{pmatrix} 1 & 1 \end{pmatrix} x(t) \end{aligned}$$

(b) **[5 Points]** The poles of the closed loop system are given by the characteristic polynomial

$$\begin{aligned}\det(A + BK - sI_2) &= (4 + k_1 - s)(4 - s) - (4 + k_2)\\ &= s^2 + s(-8 - k_1) + 12 + 4k_1 - k_2.\end{aligned}$$

By setting $\det(A + BK - sI_2) = (s + 1)(s + 2)$ and comparing the coefficients we get $k_1 = -11$ and $k_2 = -34$.

Exercise 2

1	2	3	4	Exercise
3	5	10	7	25 Points

1. The system is nonlinear, time-invariant and autonomous.
2. In state space form the system is given by

$$\dot{x}_1(t) = x_1(t)^2 x_2(t) - x_2(t)^2 + 1 \tag{2}$$
$$\dot{x}_2(t) = x_1(t) \tag{3}$$
$$y(t) = x_2(t). \tag{4}$$

3. The equilibria of the system are obtained by setting the system equations to zero

$$0 = x_1(t)^2 x_2(t) - x_2(t)^2 + 1 \tag{5}$$
$$0 = x_1(t) \tag{6}$$

such that we obtain

$$\tilde{x}^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \tilde{x}^2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{7}$$

The Jacobian is given by

$$A = \begin{pmatrix} 2\tilde{x}_1\tilde{x}_2 & -2\tilde{x}_2 \\ 1 & 0 \end{pmatrix} \tag{8}$$

and inserting the two equilibria yields

$$A_1 = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}. \tag{9}$$

A_1 has eigenvalues $\pm i\sqrt{2}$ and hence, the analysis is inconclusive. A_2 has eigenvalues $\pm\sqrt{2}$ such that we can conclude that we can conclude instability of the corresponding equilibrium.

4. The phase-plane diagram indicates sustained oscillations around the equilibrium. Looking at the linearized systems, we find that A_2 is unstable, such that trajectories starting close to the equilibrium are pushed away from it. Consequently, the phase-plane plot must correspond to the other equilibrium x^1. Additionally, A_1 has complex eigenvalues with a real part of zero, again indicating sustained oscillations. A_2 on the other hand has one positive and one negative eigenvalue, indicating a saddle point.

Exercise 3

1	2	3	Exercise
10	8	7	25 Points

1. The fact that our circuit contains two capacitors suggests using a two-dimensional state vector, with components corresponding to the voltages at C_1 and C_2 respectively, i.e.,

$$x(t) = \begin{pmatrix} u_{C1}(t) \\ u_{C2}(t) \end{pmatrix}.$$

With Kirchoff's voltage law, we can set up our first equation

$$\begin{aligned} u(t) &= u_{C1}(t) + u_{R2}(t) + u_{C2}(t) \\ &= u_{C1}(t) + R_2 C_2 \dot{u}_{C2}(t) + u_{C2}(t) \\ \dot{u}_{C2}(t) &= -\frac{1}{R_2 C_2} u_{C1}(t) - \frac{1}{R_2 C_2} u_{C2}(t) + \frac{1}{R_2 C_2} u(t). \end{aligned}$$

The second equation is obtained by applying Kirchoff's current law

$$\begin{aligned} i_{C1}(t) &= Z(t) i_{R1}(t) + i_{R2}(t) \\ C_1 \dot{u}_{C_1}(t) &= Z(t) \frac{u(t) - u_{C1}(t)}{R_1} + \frac{u(t) - u_{C1}(t) - u_{C2}(t)}{R_2} \\ \dot{u}_{C1}(t) &= -\frac{1}{C_1} \left(Z(t) \frac{1}{R_1} + \frac{1}{R_2} \right) u_{C1}(t) - \frac{1}{C_1 R_2} u_{C2}(t) + \frac{1}{C_1} \left(Z(t) \frac{1}{R_1} + \frac{1}{R_2} \right) u(t). \end{aligned}$$

The resulting two-dimensional state-space model is given by

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -\frac{1}{C_1}(Z(t)\frac{1}{R_1} + \frac{1}{R_2}) & -\frac{1}{C_1 R_2} \\ -\frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{pmatrix} x(t) + \begin{pmatrix} \frac{1}{C_1}(Z(t)\frac{1}{R_1} + \frac{1}{R_2}) \\ \frac{1}{R_2 C_2} \end{pmatrix} u(t) \\ y(t) &= (0 \quad 1) x(t) + 0 \cdot u(t). \end{aligned}$$

2. With $C_1 = C_2 = 1F$, $R_1 = R_2 = 1\Omega$ and $Z(t) = 0$ we get

$$A = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

As $D = 0$, the transfer function is given by

$$\begin{aligned} G(s) &= C(s\mathbf{I} - A)^{-1}B \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s+1 & 1 \\ 1 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & s+1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{s^2+2s} \\ &= \frac{s}{s^2+2s} = \frac{1}{s+2}. \end{aligned}$$

There is a pole-zero cancellation, hence the system is either not controllable or not observable.

3. For $Z(t) = 0$ we have $A = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$ and the characteristic polynomial of A is $\lambda^2 + 2\lambda$, hence the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = -2$. It is easy to see that the corresponding eigenvectors are $w_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Consequently, we can compute the zero input response as

$$\begin{aligned} y(t) &= C\Phi(t)x_0 = Ce^{At}x_0 = CWe^{\Lambda t}W^{-1}x_0 = \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \\ &= \frac{1}{2} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = e^{-2t}. \end{aligned}$$

Alternatively, we can also use the results from part 2 and directly obtain that

$$y(t) = L^{-1}\left\{C(s\mathbf{I} - A)^{-1}x_0\right\} = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}.$$

Exercise 4

1	2	3	Exercise
9	8	8	25 Points

1. (a) The eigenvalues are $\Lambda(\bar{A}) = \{-2, -1\}$. The corresponding eigenvectors are

$$\bar{A}w_1 = -2w_1 \quad \Rightarrow \quad w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\bar{A}w_2 = -1w_2 \quad \Rightarrow \quad w_2 = \begin{bmatrix} -6 \\ 1 \end{bmatrix}.$$

(b) Since $Re(\lambda_i) < 0$ for $i = 1, 2$ the system is asymptotically stable.

(c) In order to compute the matrix exponential it is useful to diagonalize $\bar{A}$ using the eigenvectors matrix W.

$$W = [w_1 | w_2], \quad D_{\bar{A}} = W^{-1}\bar{A}W = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$e^{D_{\bar{A}}t} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

$$e^{\bar{A}t} = We^{D_{\bar{A}}t}W^{-1} = \begin{bmatrix} e^{-2t} & 6e^{-2t} - 6e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

2. (a) According to the formula on slide 6.7:

$$A = e^{\bar{A}T} = \begin{bmatrix} e^{-2T} & 6e^{-2T} - 6e^{-T} \\ 0 & e^{-T} \end{bmatrix},$$

$$B = \int_0^T e^{\bar{A}(T-t)}\bar{B}dt = \int_0^T e^{\bar{A}(t)}\bar{B}dt = \int_0^T \begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix} dt = \begin{bmatrix} \int_0^T e^{-2t}dt \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 - e^{-2T} \\ 0 \end{bmatrix}.$$

(b) The eigenvalues of A are $\Lambda(A) = \{e^{-2T}, e^{-T}\}$. Since $|\lambda_i| < 1$ for all $T > 0$, $i = 1, 2$, the discrete time system is always asymptotically stable. This has to be the case since the free evolution of the discrete time system is just the sampled version of the free evolution of the continuous time system. Since this is asymptotically stable, for any initial condition $x(0)$,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{\bar{A}t}x(0) = 0$$

$$\Downarrow$$

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} A^k x(0) = \lim_{k\to\infty} (e^{\bar{A}T})^k x(0) = \lim_{k\to\infty} e^{\bar{A}kT}x(0) = 0.$$

3. The matrices of the new system are

$$\hat{A} = (I + \bar{A}T) = \begin{bmatrix} 1-2T & -6T \\ 0 & 1-T \end{bmatrix},$$
$$\hat{B} = \begin{bmatrix} T-T^2 \\ 0 \end{bmatrix}.$$

The eigenvalues are $\Lambda(\hat{A}) = \{1-2T, 1-T\}$. The system is asymptotically stable iff

$$|1-2T| < 1 \quad \Rightarrow \quad T < 1 \quad \& \quad |1-T| < 1 \quad \Rightarrow \quad T < 2.$$

Therefore the condition is $T \geq 1$.