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D-ITET
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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	Exercise
7	6	5	7	25 Points

1. The controllability matrix $P \in \mathbb{R}^{2\times 4}$ of system (3) is given by

$$\begin{aligned} P &= \begin{bmatrix} [b_1\ b_2] & A[b_1\ b_2] \end{bmatrix} \\ &= \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 \end{bmatrix}. \end{aligned}$$

For system (3) to be controllable matrix P should be of full rank, i.e. $rank(P) = 2$. Assume that (1) is controllable. Its controllability matrix $P_1 = \begin{bmatrix} b_1 & Ab_1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$ would then have full rank. Therefore, its columns b_1, Ab_1 are linearly independent. The latter implies that the first and third column of P are linearly independent, hence P has full rank and (3) is controllable irrespective of the fact that (2) may be uncontrollable. The same conclusions is drawn if (2) was assumed to be controllable instead.

Alternative solution: by setting $u_2(t) = 0$, (3) turns into (1), which is controllable. Hence, (3) is controllable. Similarly for the case where (2) is controllable.

2. Consider the controllability matrices $P_1, P_2 \in \mathbb{R}^{2\times 2}$ of systems (1) and (2), respectively.

$$\begin{aligned} P_1 &= \begin{bmatrix} b_1 & Ab_1 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} b_2 & Ab_2 \end{bmatrix}. \end{aligned}$$

In the case where (1), (2) are not controllable, the columns of P_1 and P_2 are linearly dependent. Therefore, there exist scalars $k_1, k_2 \in \mathbb{R}$ such that $Ab_1 = k_1 b_1$ and $Ab_2 = k_2 b_2$.

As shown in part 1, the controllability matrix $P \in \mathbb{R}^{2\times 4}$ of system (3) is given by $P = \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 \end{bmatrix}$. We thus have that $P = \begin{bmatrix} b_1 & b_2 & k_1 b_1 & k_2 b_2 \end{bmatrix}$. For system (3) to be controllable, matrix P should have full rank, i.e. $rank(P) = 2$. The latter holds if b_1 and b_2 are linearly independent.

3. Consider for example the system matrices $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 0 & -1 \end{bmatrix}$. Under this choice both systems (1) and (2) are observable, since the observability matrices Q_1 and Q_2 have full rank.

$$\begin{aligned} Q_1 &= \begin{bmatrix} C_1 \\ C_1 A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow rank(Q_1) = 2, \\ Q_2 &= \begin{bmatrix} C_2 \\ C_2 A \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \Rightarrow rank(Q_2) = 2. \end{aligned}$$

However, system (3) is not observable since its observability matrix does not have full rank.

$$Q = \begin{bmatrix} (C_1 + C_2) \\ (C_1 + C_2)A \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow rank(Q) = 1.$$

Other counterexamples are of course equally valid.

4. Applying $u_1(t) = Kx_1(t) + r(t)$ to (1) the closed loop system is

$$\begin{aligned}\dot{x}_1(t) &= Ax_1(t) + b_1 K x_1(t) + b_1 r(t),\\ &= (A + b_1 K)x_1(t) + b_1 r(t).\end{aligned}$$

The controllability matrix of the closed loop system is given by

$$\begin{aligned}P &= \begin{bmatrix} b_1 & (A + b_1 K)b_1 \end{bmatrix}\\ &= \begin{bmatrix} b_1 & Ab_1 + b_1 K b_1 \end{bmatrix}.\end{aligned}$$

Since Kb_1 is a scalar the above statement can be written as

$$P = \begin{bmatrix} b_1 & Ab_1 + \alpha b_1 \end{bmatrix},$$

where $\alpha = Kb_1 \in \mathbb{R}$. P has full rank (i.e. the closed loop system is controllable) if and only if the columns b_1 and $Ab_1 + \alpha b_1$ are linearly independent. The latter is equivalent with the requirement that b_1 and Ab_1 are linearly independent. But this is the case if and only if the controllability matrix of (1) is full rank (i.e. (1) is controllable). Therefore, the closed loop system is controllable from $r(t)$ if and only if (1) is controllable from $u_1(t)$.

Exercise 2

1	2	Exercise
10	15	25 Points

1. (a) **[3 Points]**

$$x_1[k+1] = e^{-T}x_1[k] + \underbrace{\int_0^T e^{\tau-T}\mathrm{d}\tau}_{1-e^{-T}}\, u[k]$$

$$y[k] = x_1[k]$$

(b) **[3 Points]**

$$x_2[k+1] = e^{-T}x_2[k] + \underbrace{\int_0^T e^{\tau-T}\mathrm{d}\tau}_{1-e^{-T}}\, y[k]$$

$$v[k] = -2x_2[k]$$

(c) **[4 Points]** Define $\xi[k] := \begin{bmatrix} x_1[k] & x_2[k] \end{bmatrix}^\top$.

$$\begin{aligned}
\xi[k+1] &= \begin{pmatrix} x_1[k+1] \\ x_2[k+1] \end{pmatrix} = \begin{pmatrix} e^{-T}x_1[k] + (1-e^{-T})u[k] \\ e^{-T}x_2[k] + (1-e^{-T})y[k] \end{pmatrix} = \begin{pmatrix} e^{-T}x_1[k] + (1-e^{-T})u[k] \\ e^{-T}x_2[k] + (1-e^{-T})x_1[k] \end{pmatrix} \\
&= \begin{pmatrix} e^{-T} & 0 \\ 1-e^{-T} & e^{-T} \end{pmatrix} \begin{pmatrix} x_1[k] \\ x_2[k] \end{pmatrix} + \begin{pmatrix} 1-e^{-T} \\ 0 \end{pmatrix} u[k] \\
&= \begin{pmatrix} e^{-T} & 0 \\ 1-e^{-T} & e^{-T} \end{pmatrix} \xi_k + \begin{pmatrix} 1-e^{-T} \\ 0 \end{pmatrix} u[k] \\
v[k] &= \begin{pmatrix} 0 & -2 \end{pmatrix} \begin{pmatrix} x_1[k] \\ x_2[k] \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2 \end{pmatrix} \xi[k]
\end{aligned}$$

2. (a) **[3 Points]** Define $\xi(t) := \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^\top$.

$$\begin{aligned}
\dot{\xi} &= \begin{pmatrix} \dot{x}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -x_1 + u \\ -x_2 + y \end{pmatrix} = \begin{pmatrix} -x_1 + u \\ -x_2 + x_1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\
v &= -2x_2 = \begin{pmatrix} 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{aligned}$$

(b) **[1+5+5+1 Points]** Recall $E = e^{\bar{E}T}, \quad F = \int_0^T e^{\bar{E}(T-\tau)}\bar{F}\mathrm{d}\tau, \quad G = \begin{bmatrix} 0 & -2 \end{bmatrix}$
[1 Point for G]

$$\bar{E} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{R_1} + \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{R_2}$$

Since R_1 is diagonal R_1 and R_2 commute. Therefore

$$e^{\bar{E}T} = e^{(R_1+R_2)T} = e^{R_1 T} e^{R_2 T},$$

where

$$\begin{aligned} e^{R_1 T} &= \begin{pmatrix} e^{-T} & 0 \\ 0 & e^{-T} \end{pmatrix} \\ e^{R_2 T} &= I + TR_2 + \underbrace{\frac{T^2}{2} R_2^2 + \dots}_{0} = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}. \end{aligned}$$

Hence

$$E = e^{\bar{E}T} = \begin{pmatrix} e^{-T} & 0 \\ Te^{-T} & e^{-T} \end{pmatrix}.\textbf{[5 Points for E]}$$

$$\begin{aligned} F &= \int_0^T \begin{pmatrix} e^{\tau-T} & 0 \\ (T-\tau)e^{\tau-T} & e^{\tau-T} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\tau = \int_0^T \begin{pmatrix} e^{\tau-T} \\ (T-\tau)e^{\tau-T} \end{pmatrix} \mathrm{d}\tau \\ &= \begin{pmatrix} 1-e^{-T} \\ 1-e^{-T}(1+T) \end{pmatrix}, \end{aligned}$$

where we have used that

$$\begin{aligned} \int_0^T e^{\tau-T} \mathrm{d}\tau &= e^{\tau-T}\Big|_0^T = 1-e^{-T} \\ \int_0^T \tau e^{\tau-T} \mathrm{d}\tau &\overset{P.I.}{=} \tau e^{\tau-T}\Big|_0^T - \int_0^T e^{\tau-T} \mathrm{d}\tau \\ &= T-(1-e^{-T}) = T-1+e^{-T}.\textbf{[5 Points for F]} \end{aligned}$$

Finally we have for $\xi[k] := (x_1[k], x_2[k])^T$

$$\begin{aligned} \xi[k+1] &= E\xi[k] + Fu[k] \\ v[k] &= G\xi[k] \end{aligned}$$

[1 Point]

Exercise 3

1	2	3	4	5	Exercise
5	5	7	4	4	25 Points

1. The state equations for the system are:

$$\dot{x} = \begin{bmatrix} -1 & -1 \\ \alpha - 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} -3 & -1 \end{bmatrix} x$$

2. The controllability matrix is $P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & \alpha - 1 \end{bmatrix}$. Hence, $det(P) = \alpha - 1$ and the system is controllable unless $\alpha = 1$.

   For $\alpha = 1$ the reachable states are $range(P) = span\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

3. For $\alpha = -2$ the transfer function is given by

$$G(s) = C\,(sI - A)^{-1}\,B = \begin{bmatrix} -3 & -1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ 3 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \frac{1}{s^2 - 1 - 3} \begin{bmatrix} -3 & -1 \end{bmatrix} \begin{bmatrix} s-1 \\ -3 \end{bmatrix}$$
$$= \frac{-3(s-1)+3}{s^2-4} = \frac{-3s+6}{(s+2)(s-2)} = \frac{-3(s-2)}{(s+2)(s-2)} = \frac{-3}{s+2}.$$

   There is a pole-zero cancellation at $s = 2$, hence the system is either not controllable or not observable. In part 2 of the exercise it was shown that the system is controllable for $\alpha \neq 1$, hence for $\alpha = -2$ it is controllable and thus cannot be observable.

4. $y(t) = 1 \;\; \forall\, t \geq 1$ holds if and only if $Cx(t) = -3x_1(t) - x_2(t) = 1 \;\; \forall\, t \geq 1$. Hence, the set of all states for which $y(t) = 1$ is the space $V = \left\{x \in \mathbb{R}^2 : x_2 = -3x_1 - 1\right\}$. The equilibrium states when $u(t) = 0 \;\; \forall\, t \geq 0$ are the states $x \in \mathbb{R}^2$ with $\dot{x} = 0$, which are the states in the nullspace of A: $x \in Null(A) = span\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

5. With $u(t) = 0 \;\; \forall t \;\geq 1$, since the only eigenvector of A is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $y(t) = 1 \;\; \forall\, t \geq 1$ is only possible if $\dot{x} = 0 \;\; \forall\, t \geq 1$, hence $x \in Null(A) = span\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$. Thus, the state has to lie in the intersection $V \cap Null(A) = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

   Since for $\alpha = 2$ the system is controllable it can be driven to $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ in an arbitrary time interval $[0,\; t_1]$. With $u(t) = 0 \;\; \forall\, t > t_1$ it will stay in this point and all the requirements are satisfied.

Exercise 4

1	2	3	4	Exercise
3	6	13	3	25 Points

1. The system is non-linear and autonomous but time invariant.

2. In state space form, the system is given by

$$\dot{x}_1(t) = f_1(x(t), u(t)) = -2x_1^3(t) - x_2^2(t) + 2x_2(t) - 1 + c^2 \tag{1}$$
$$\dot{x}_2(t) = f_2(x(t), u(t)) = x_1(t) \tag{2}$$

and

$$y = g(x(t), u(t)) = x_2(t) \tag{3}$$

The dimension of the system is 2 (number of states).

3. The equilibria are obtained by setting the state derivatives to zero, i.e.,

$$0 \equiv -2x_1^3(t) - x_2^2(t) + 2x_2(t) - 1 + c^2 \tag{4}$$
$$0 \equiv x_1(t) \tag{5}$$

and we obtain

$$\tilde{x}^{(1)} = \begin{bmatrix} 0 \\ 1-c \end{bmatrix} \quad \text{and} \quad \tilde{x}^{(2)} = \begin{bmatrix} 0 \\ 1+c \end{bmatrix}. \tag{6}$$

The system matrix of the linearized model for each of the equilibria is determined by the partial derivatives of the system equations, i.e.,

$$A^{(i)} = \begin{bmatrix} -6x_1^2(t) & -2x_2(t)+2 \\ 1 & 0 \end{bmatrix}_{x(t)=\tilde{x}^{(i)}} \tag{7}$$

for $i = 1, 2$. With the equilibria given above, we further obtain

$$A^{(1)} = \begin{bmatrix} 0 & 2c \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad A^{(2)} = \begin{bmatrix} 0 & -2c \\ 1 & 0 \end{bmatrix}. \tag{8}$$

The eigenvalues of $A^{(1)}$ are given by

$$\lambda_1^{(1)} = \sqrt{2c} = 2 \quad \text{and} \quad \lambda_2^{(1)} = -\sqrt{2c} = -2. \tag{9}$$

Since one eigenvalue is greater than zero, the equilibrium $\tilde{x}^{(1)}$ is unstable.

Matrix $A^{(2)}$ has eigenvalues

$$\lambda_1^{(2)} = \sqrt{-2c} = 2i \quad \text{and} \quad \lambda_2^{(2)} = -\sqrt{-2c} = -2i. \tag{10}$$

Since the real parts of the eigenvalues are zero, the analysis is inconclusive for equilibrium $\tilde{x}^{(2)}$.

4. The stability analysis using linearization is always inconclusive for the equilibrium $\tilde{x}^{(2)}$ since for every c, the real parts of the corresponding eigenvalues are zero. For the case $c = 0$, also the eigenvalues of the first equilibrium are zero.