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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	Exercise
10	9	8	27 Points

1. The fact that our circuit contains two capacitors suggests using a two-dimensional state vector, with it's components corresponding to the voltages at C_1 and C_2 respectively, i.e.,

$$x(t) = \begin{pmatrix} u_{C1} \\ u_{C2} \end{pmatrix}.$$

with Kirchoff's voltage law, we can set up our first equation

$$\begin{aligned} u(t) &= u_{C1}(t) + u_{R2}(t) + u_{C2}(t) \\ &= u_{C1}(t) + R_2 C_2 \dot{u}_{C2}(t) + u_{C2}(t) \\ \dot{u}_{C2}(t) &= -\frac{1}{R_2 C_2} u_{C1}(t) - \frac{1}{R_2 C_2} u_{C2}(t) + \frac{1}{R_2 C_2} u(t). \end{aligned} \tag{1}$$

The second equation is obtained by applying Kirchoff's current law

$$\begin{aligned} i(t) &= i_{R1}(t) + i_{R2}(t) \\ C_1 \dot{u}_{C_1}(t) &= \frac{u(t) - u_{C1}(t)}{R_1} + \frac{u(t) - u_{C1}(t) - u_{C2}(t)}{R_2} \\ \dot{u}_{C1}(t) &= -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) u_{C1}(t) - \frac{1}{C_1 R_2} u_{C2}(t) + \frac{1}{C_1}(\frac{1}{R_1} + \frac{1}{R_2}) u(t). \end{aligned}$$

The resulting two-dimensional state-space model is given by

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -\frac{1}{C_1}(\frac{1}{R_1} + \frac{1}{R_2}) & -\frac{1}{C_1 R_2} \\ -\frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{pmatrix} x(t) + \begin{pmatrix} \frac{1}{C_1}(\frac{1}{R_1} + \frac{1}{R_2}) \\ \frac{1}{R_2 C_2} \end{pmatrix} u(t) \\ y(t) &= (0 \quad 1) x(t) + 0 \cdot u(t). \end{aligned}$$

2. With $C_1 = C_2 = 1F$ and $R_1 = R_2 = 1\Omega$ we get

$$A = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

As $D = 0$, the transfer function is given by

$$\begin{aligned}
G(s) &= C(s\mathbf{I} - A)^{-1}B \\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s+2 & 1 \\ 1 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & s+2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \frac{1}{(s+2)(s+1)-1} \\
&= \frac{s}{s^2+3s+1}.
\end{aligned}$$

3. In this case, we can simply evaluate the transfer function at the frequency of the input sine signal to calculate the steady state output signal as

$$y(t) = |G(j\omega)|U_0 sin(\omega t + \phi_0 + \angle G(j\omega)).$$

With $U_0 = 1$ and $\phi_0 = 0^\circ$ we obtain

$$\begin{aligned}
y(t) &= |G(j1)|sin(1t + \angle G(j1)) \\
|G(j1)| &= \frac{|j|}{|-1+3j+1|} = \frac{1}{3} \\
\angle G(j1) &= \arctan\left(\frac{0}{\frac{1}{3}}\right) = 0 \\
\rightarrow y(t) &= \frac{1}{3} sin(t).
\end{aligned}$$

Exercise 2

1	2	3	Exercise
7	9	10	26 Points

1. Consider the discrete time system

$$\begin{aligned} &z(k+2) - 0.7z(k+1) + 0.1z(k) = 0.5u(k), \\ &y(k) = z(k+1). \end{aligned} \tag{2}$$

Let $x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} z(k+1) \\ z(k) \end{bmatrix}$. Hence, we have $x_2(k+1) = z(k+1) = x_1(k)$, and $x_1(k+1) = z(k+2)$. By inspection of (2)

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.7 & -0.1 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k). \end{aligned} \tag{3}$$

Denote then $A = \begin{bmatrix} 0.7 & -0.1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $D = 0$.

2. i) Consider the determinant of $\lambda I - A$, where λ represents the eigenvalues of the system. Then, $\det(\lambda I - A) = \lambda^2 - 0.7\lambda + 0.1$. By equating with zero we get the eigenvalues of the system, which are $\lambda_1 = 0.5$ and $\lambda_2 = 0.2$. Since, $|\lambda_i| < 1$ for $i = 1, 2$, the system is asymptotically stable.
ii) Compute the controllability matrix P

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0.5 & 0.35 \\ 0 & 0.5 \end{bmatrix}.$$

P is full rank ($Rank(P) = 2$), so the system is controllable.
iii) Compute the observability matrix Q

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.7 & -0.1 \end{bmatrix}.$$

Q is full rank ($Rank(Q) = 2$), so the system is observable.

3. The observer dynamics are given by

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + L(y(k) - \hat{y}(k)) + Bu(k), \\ \hat{y}(k) &= C\hat{x}(k). \end{aligned}$$

By subtracting (3) from the previous equation, and since $e(k+1) = \hat{x}(k+1) - x(k+1)$,

$$e(k+1) = (A - LC)e(k). \tag{4}$$

Since we want the eigenvalues of the observation error $e(k)$ to be both 0.1, we can design a gain matrix $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \in \mathbb{R}^2$ such that the eigenvalues of $A - LC$ are both at 0.1. Hence,

$$\det\left(\begin{bmatrix} \lambda - (0.7 - l_1) & 0.1 \\ l_2 - 1 & \lambda \end{bmatrix}\right) = \lambda^2 - (0.7 - l_1)\lambda + 0.1(1 - l_2). \tag{5}$$

By equating the last equation with zero, and since both eigenvalues are equal, we have that

$$\lambda_1 = \lambda_2 = \frac{0.7 - l_1}{2} = 0.1.$$

Hence, $l_1 = 0.5$. Due to the fact that $\lambda_1 = \lambda_2$, the last term of (5) must be $0.1(1 - l_2) = 0.1^2$. Therefore, $l_2 = 0.9$.

Exercise 3

1	2	3	4	Exercise
5	7	9	6	27 Points

1. The state space representation of the system can be given as

$$\dot{x_1}(t) = \frac{x_2(t)}{x_2(t)^2+1}x_1(t)$$
$$\dot{x_2}(t) = -\frac{x_2(t)}{x_2(t)^2+1}x_1(t)$$

where $x_1(t) = y(t)$ and $x_2(t) = z(t)$.

The dimension of the system is 2. The system is autonomous since it has no input. The system is nonlinear.

2. There are an infinite number of equilibria. Specifically, any point on the line $y = 0$ is an equilibrium. Likewise, any point on the line $z = 0$ is also an equilibrium.

3. Linearizing the system about $y = \hat{y}$ and $z = \hat{z}$, we obtain the Jacobian matrix

$$A = \begin{bmatrix} \frac{\hat{z}}{\hat{z}^2+1} & -\frac{\hat{z}^2-1}{(\hat{z}^2+1)^2}\hat{y} \\ -\frac{\hat{z}}{\hat{z}^2+1} & \frac{\hat{z}^2-1}{(\hat{z}^2+1)^2}\hat{y} \end{bmatrix}.$$

If $\hat{y} > 0$ (or $\hat{y} < 0$) then for $(\hat{y}, \hat{z})$ to be an equilibrium $\hat{z} = 0$. The Jacobian matrix reduces to

$$A = \begin{bmatrix} 0 & \hat{y} \\ 0 & -\hat{y} \end{bmatrix}..$$

The eigenvalues for A are $\lambda = 0$ and $\lambda = -\hat{y}$. Therefore, if $\hat{y} > 0$ we cannot evaluate the stability of the system through linearization since one of the eigenvalues is zero. In case the system has $\hat{y} < 0$, the system is unstable due to a positive eigenvalue.

If $\hat{z} > 0$ (or $\hat{z} < 0$) then for $(\hat{y}, \hat{z})$ to be an equilibrium $\hat{y} = 0$. The Jacobian matrix reduces to

$$A = \begin{bmatrix} \frac{\hat{z}}{\hat{z}^2+1} & 0 \\ -\frac{\hat{z}}{\hat{z}^2+1} & 0 \end{bmatrix}.$$

The eigenvalues for A are $\lambda = 0$ and $\lambda = \frac{\hat{z}}{\hat{z}^2+1}$. Therefore, if $\hat{z} < 0$ we cannot evaluate the stability of the system through linearization since one of the eigenvalues is zero. In case the system has $\hat{z} > 0$, the system is unstable due to a positive eigenvalue.

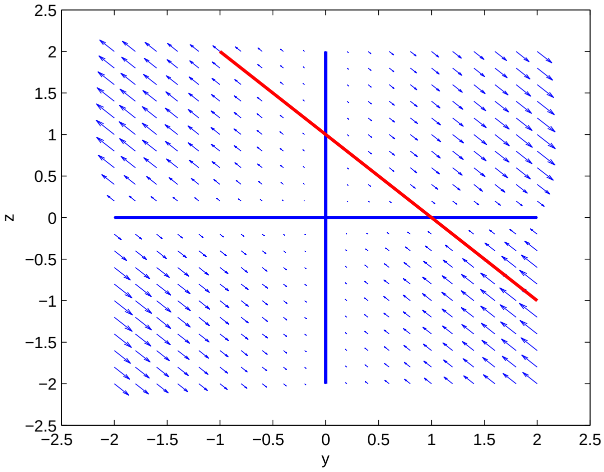

Figure 1: Plot of vector field, invariant set $y + z = 1$, and all equilibria for Exercise 3.

4. Consider the function

$$V(y, z) = y + z.$$

Differentiating V by time, we have that

$$\dot{V}(y, z) = \frac{dV}{dy}\dot{y} + \frac{dV}{dz}\dot{z} \qquad = 0.$$

Therefore, if $V(y, z) = y+z = c$, then $y+z = c$ always since $\dot{V}(y, z) = 0$ independent of y, z, and $c \in \mathbb{R}$.

All equilibria and the invariant set corresponding to $c = 1$ are illustrated in Figure 1. According to the Figure, the invariant line $y + z = 1$ goes through two equilibrium points, $(y, z) = (1, 0)$ and $(y, z) = (0, 1)$. According to part 3 above, we know that the equilibrium $\hat{z} > 0$ is unstable, therefore we can expect the system to move away from equilibrium point $(y, z) = (0, 1)$. If the system starts at the point (y, z) where $y + z = 1$, $y > 0$, and $z > 0$, then the system will move along the line away from $(y, z) = (0, 1)$ and converge at $(y, z) = (1, 0)$. When $y > 0$ and $z < 0$, it always holds that $\dot{y} < 0$ and $\dot{z} > 0$, therefore when the system starts at (y, z), $y > 0$ and $z < 0$, we expect the system to converge at $(y, z) = (1, 0)$.

Exercise 4

1	2	3	Exercise
6	4	10	20 Points

1. Assuming that the matrix A is *diagonalizable*, one can use the matrix of eigenvectors W to induce a change of coordinates:

$$A = W\Lambda W^{-1}$$

with which the state transition matrix can be represented by:

$$\Phi(t) = e^{At} = We^{\Lambda t}W^{-1}.$$

Therefore one can read of the eigenvalues directly from:

$$e^{\Lambda t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The eigenvalue obtained are $\lambda_1 = -1$ and $\lambda_2 = 2$, which makes the system *unstable* since $\text{Re}[\lambda_2] > 0$. Clearly the system can not be asymptotically stable.

2. The eigenvectors have to be linearly independent, since the system has distinct eigenvalues λ_1 and λ_2.

3. The time derviative of Φ evaluated at time $t = 0$ is

$$\frac{d}{dt}\Phi(t)_{|t=0} = Ae^{At}{}_{|t=0} = A.$$

Taking the derivative of the given transition matrix, we obtain:

$$\frac{d}{dt}\Phi(t) = \frac{1}{3}\begin{bmatrix} -2e^{-t} + 2e^{2t} & 2e^{-t} + 4e^{2t} \\ e^{-t} + 2e^{2t} & -e^{-t} + 4e^{2t} \end{bmatrix},$$

and evaluating at $t = 0$:

$$A = \frac{1}{3}\begin{bmatrix} -2+2 & 2+4 \\ 1+2 & -1+4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}.$$