Automatic Control Laboratory
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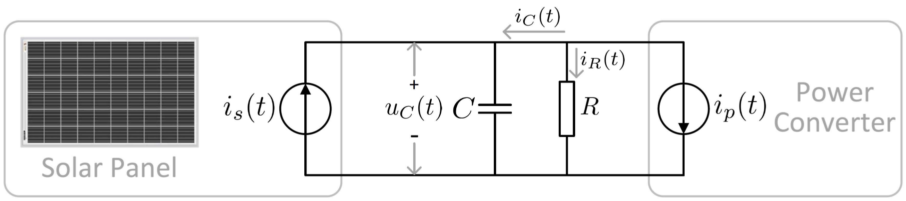
Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	Exercise
3	5	5	4	8	25 Points



1. The dynamic equation of the capacitor is

$$C\frac{du_C(t)}{dt} = i_s(t) + i_C(t)\,.(\mathbf{1\ p})$$

The equation of the resistor is

$$i_R(t) = \frac{u_C(t)}{R}\,.(\mathbf{1\ p})$$

It follows from Kirchhoff's law that

$$i_C(t) + i_R(t) + i_p(t) = 0\,.(\mathbf{1\ p})$$

By combining the above equations, we have

$$C\frac{du_C(t)}{dt} = i_s(t) - i_p(t) - \frac{u_C(t)}{R}\,,$$

which can be reformulated as the claimed result. (If the students directly give this equation with some explanation on the circuit, can also give 2 points.)

2. The controller $G(s) = K_\mathrm{P} + \frac{K_\mathrm{I}}{s}$ includes a proportional term and an integral action, and the control law in frequency domain is

$$L\{i_p(t)\} = \left(K_\mathrm{P} + \frac{K_\mathrm{I}}{s}\right) L\{u_C(t) - u_\mathrm{ref}(t)\}\,, (\mathbf{1\ p})$$

and the time-domain expression is

$$i_p(t) = \underbrace{K_\mathrm{P}(u_C(t) - u_\mathrm{ref}(t))}_{(\mathbf{1\ p})} + \underbrace{K_\mathrm{I}\int_0^t u_C(\tau) - u_\mathrm{ref}(\tau)d\tau}_{(\mathbf{1\ p})}\,.$$

Let $z(t) = \int_0^t u_C(\tau) - u_\mathrm{ref}(\tau)d\tau$ **(1 p)** plays the role of the controller state such that the equation above can be reformulated as

$$\dot{z}(t) = u_C(t) - u_\mathrm{ref}(t)(\mathbf{1\ p})$$
$$i_p(t) = K_\mathrm{P}(u_C(t) - u_\mathrm{ref}(t)) + K_\mathrm{I}z(t)\,.$$

3. It follows from the equations in part 1 and part 2 that

$$\frac{d}{dt}\begin{bmatrix} u_C(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} - \frac{K_\mathrm{P}}{C} & -\frac{K_\mathrm{I}}{C} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_C(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} \frac{K_\mathrm{P}}{C} & \frac{1}{C} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_\mathrm{ref}(t) \\ i_s(t) \end{bmatrix},$$

$$u_C(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} u_C(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} u_\mathrm{ref}(t) \\ i_s(t) \end{bmatrix}.$$

Hence,

$$A = \begin{bmatrix} -\frac{1}{RC} - \frac{K_\mathrm{P}}{C} & -\frac{K_\mathrm{I}}{C} \\ 1 & 0 \end{bmatrix} (\mathbf{2\ p}), B = \begin{bmatrix} \frac{K_\mathrm{P}}{C} & \frac{1}{C} \\ -1 & 0 \end{bmatrix} (\mathbf{1\ p}), C = \begin{bmatrix} 1 & 0 \end{bmatrix} (\mathbf{1\ p}), D = \begin{bmatrix} 0 & 0 \end{bmatrix} (\mathbf{1\ p}).$$

4. The equilibrium of the system $\hat{x}$ satisfies $\dot{x}(t) = 0$ **(1 p)**. Hence, we have

$$\begin{bmatrix} -\frac{1}{RC} - \frac{K_\mathrm{P}}{C} & -\frac{K_\mathrm{I}}{C} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_C \\ \hat{z} \end{bmatrix} + \begin{bmatrix} \frac{K_\mathrm{P}}{C} & \frac{1}{C} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ i_0 \end{bmatrix} = 0\ (\mathbf{1\ p}),$$

which leads to

$$\hat{x} = \begin{bmatrix} \hat{u}_C \\ \hat{z} \end{bmatrix} = \begin{bmatrix} u_0 \\ \frac{i_0}{K_\mathrm{I}} - \frac{u_0}{RK_\mathrm{I}} \end{bmatrix} (\mathbf{2\ p}).$$

5. The characteristic polynomial of the system is

$$\det(sI_2 - A) = \underbrace{s^2}_{(\mathbf{1\ p})} + \underbrace{\left(\frac{1}{RC} + \frac{K_\mathrm{P}}{C}\right) s}_{(\mathbf{1\ p})} + \underbrace{\frac{K_\mathrm{I}}{C}}_{(\mathbf{1\ p})}.$$

The system is asymptotically stable if and only if the real parts of the roots are negative **(1 p)**, which requires $\frac{1}{RC} + \frac{K_\mathrm{P}}{C} > 0$ **(1 p)** and $\frac{K_\mathrm{I}}{C} > 0$ **(1 p)**. Since $C > 0$ and $R > 0$, the conditions for K_P and K_I are $K_\mathrm{I} > 0$ **(1 p)** and $K_\mathrm{P} > -\frac{1}{R}$ **(1 p)**.

Exercise 2

1	2	3	4	5	Exercise
4	**4**	**4**	**7**	**6**	**25 Points**

1. It follows from the block diagram that

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & \alpha-3 & 0 \\ 1 & \alpha & \alpha-1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} u(t)\,,$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}\,.$$

   Hence,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & \alpha-3 & 0 \\ 1 & \alpha & \alpha-1 \end{bmatrix} (\mathbf{2\ p})\, B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} (\mathbf{1\ p})$$

$$C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} (\mathbf{1\ p})\,, D = 0\ \ .$$

2. To answer the question we have to analyse the eigenvalues of the matrix A. Notice that A is a triangular matrix, therefore its eigenvalues correspond to the diagonal entries. So, $\text{eig}(A) = \{-1, \alpha-3, \alpha-1\}$ **(2 p)** . For a system to be asympotically stable we must have $\lambda_i < 0$ for all i. Therefore the system is asympotically stable if and only if $\alpha - 3 < 0$ (i.e. $\alpha < 3$) and $\alpha - 1 < 0$ (i.e. $\alpha < 1$) **(1 p)** . Combining these two conditions, we have that the system is asympotically stable iff $\alpha < 1$ **(1 p)** .

3. The observability matrix reads as,

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

   Therefore:

$$O = \begin{bmatrix} 1 & 0 & \gamma \\ -4 & 1 & 0 \\ 16 & -5 & 2 \end{bmatrix} (\mathbf{1\ p})$$

   An easy computation shows that:

$$\det(O) = \det\left(\begin{bmatrix} 1 & 0 & \gamma \\ -4 & 1 & 0 \\ 16 & -5 & 2 \end{bmatrix}\right) = 4\gamma + 2(\mathbf{1\ p})$$

   We know that if the $\det(O) = 0$, then matrix O has not full rank and thus the system is not observable **(1 p)**. Therefore, the system is observable $\forall \gamma \in \mathbb{R} \setminus \{-\frac{1}{2}\}$ **(1 p)**.

4. The set of reachable states X is given by the image of the controllability matrix. The controllability matrix is given by P $= \begin{bmatrix} B & AB & A^2B \end{bmatrix}$.

   For $\gamma = 0$, we have B $= \begin{bmatrix} 0 & 0 & \beta \end{bmatrix}^T$ and thus

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2\beta \\ 0 & 2\beta & -2\beta \\ \beta & 0 & 0 \end{bmatrix} \textbf{(1 p)}$$

   Therefore, in this case :

$$X = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{s.t.} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \text{Im} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & -2 \\ 1 & 0 & 0 \end{bmatrix} \right\} \textbf{(1 p)}$$

   For $\beta = 0$, we have B $= \begin{bmatrix} \gamma & 0 & 0 \end{bmatrix}^T$ and thus

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} \gamma & -4\gamma & 16\gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \textbf{(1 p)}$$

   Therefore, in this case:

$$X = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{s.t.} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \text{Im} \begin{bmatrix} 1 & -4 & 16 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} = \text{Im} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \textbf{(1 p)}$$

   To argue which is the better choice, we can observe that for the first case (i.e. for $\gamma = 0$), we have $\det(P) = -4\beta^3$, therefore the controllability matrix P has full rank for $\beta \neq 0$. Therefore, if we choose $\gamma = 0$, the system is controllable for $\beta \neq 0$ and the whole space is reachable**(1 p)**.
   Instead, for the second case ($\beta = 0$) the controllability matrix P has clearly not full rank for any value of γ. Therefore, if we choose $\beta = 0$, then the system is never controllable and only states on the form $[a, 0, 0]^T$ for $a \in \mathbb{R}$ can be reached **(1 p)**.
   Therefore the choice $\gamma = 0$ and $\beta \neq 0$ is a better one for controllability purposes **(1 p)**.
   (*Note: Full points when both reachability and controllability are used in the discussion, -1 (i.e. maximum is 6) if only one of the two is used.*

5. First, we derive the closed-loop state dynamics as $\dot{x} = \tilde{A}x$ where:

$$\tilde{A} = A + BK = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \textbf{(1 p)}$$

   Therefore:

$$\tilde{A} = \begin{bmatrix} k_1 - 4 & k_2 + 1 & k_3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \textbf{(1 p)}$$

We recognize that $\tilde{A}$ is a triangular matrix, therefore its eigenvalues correspond to the diagonal entries: $\text{eig}(A) = \{k_1 - 4, -1, 0\}$ **(1 p)**.
In order to place the poles in the given positions, we have to enforce the eigenvalues of $\tilde{A}$ to be exactly equal to the wanted poles. We see that two eigenvalues already match the required position and the only one left to be placed is $k_1 - 4$, which is placed at -5 iff $k_1 - 4 = -5$, i.e. $k_1 = -1$ **(2 p)**. Note that, since all remaining parameters k_2 and k_3 are not assigned, i.e. they are free, there exists infinitely many matrices K satisfying the given requirement of poles placement.

As the closed-loop system will have eigenvalues placed at -5, -1 and 0, it will be stable as $\lambda_i \leq 0 \quad \forall\, i = 1, 2, 3$ (but it will be not asympotically stable due to the 0 eigenvalue) **(1 p)**.

Exercise 3

1 (a)	1 (b)	1 (c)	1(d)	2 (a)	2 (b)	2 (c)	Exercise
8	**3**	**4**	**4**	**2**	**2**	**2**	**25 Points**

1. (a) Since the systems are controllable and observable, we can use their transfer functions to infer their stability properties. In this scenario there are indeed no cancellations happening because all modes are observable and controllable.

   i. $p_{1,2} = -10$, $p_{3,4} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ **(1 pt)**. The system is asymptotically stable because all poles have negative real part **(1 pt)**.

   ii. $p_1 = -10$, $p_2 = 10$, $p_{3,4} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ **(1 pt)**. The system is unstable because it has a pole with positive real part **(1 pt)**.

   iii. $p_{1,2} = 0$, $p_3 = -10$, $p_{4,5} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ **(1 pt)**. Because of the repeated pole in zero and the other poles have negative real part, we can not conclude if the system is marginally stable or unstable just by looking at the poles/eigenvalues but we would need to study the eigenvectors to draw such conclusion **(1 pt)**.

   iv. $p_1 = 0$, $p_2 = -10$, $p_{3,4} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ **(1 pt)**. The system is marginally stable because of the pole in zero and the other poles have negative real part **(1 pt)**.

   (b) We apply the Nyquist stability criterion. We know that the closed-loop system is stable if and only if $N = -P$, where N is the number of times the Nyquist plot encircles $(-1/K, 0)$ in the clockwise direction, and P is the number of poles of $G_1(s)$ with positive real parts. Since $P = 0$, N must be zero to have closed-loop stability **(1 pt)**. When $K = \frac{1}{5}$, $N = 1$ therefore the closed-loop is not stable **(1 pt)**. When $K = -\frac{1}{5}$, then $N = -P = 0$, therefore the closed-loop system is stable **(1 pt)**.

   (c) The system with $G_1(s)$ as transfer function is asymptotically stable, therefore the steady-state response for a sinusoidal input can be computed using the following formula:

$$y(t) = 2|G_1(j10)| \sin(10t + \angle G_1(j10)) \quad \textbf{(1 pt)}. \tag{1}$$

$$\begin{aligned} G_1(j10) &= \frac{-10^3}{(j10+10)^2(-99+j10)} \\ &= \frac{-10^3}{(2 \cdot 10^2)j(-99+j10)} \\ &= \frac{-5}{-10-j99} \\ &= \frac{5}{10+j99}. \end{aligned} \tag{2}$$

$$
\begin{aligned}
|G_1(j10)| &= \left|\frac{5}{10+j99}\right| \\
&= \frac{5}{\sqrt{10^2+99^2}} \\
&\approx \frac{5}{99} \quad \textbf{(1 pt)}\,.
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\angle G_1(j10) &= \tan^{-1}\left(\frac{0}{5}\right) - \tan^{-1}\left(\frac{99}{10}\right) \\
&= -1.47 \text{ rad} \\
&\approx -84^\circ \quad \textbf{(1 pt)}\,.
\end{aligned} \tag{4}
$$

Alternatively, the phase can also be computed as follow

$$
\begin{aligned}
\angle G_1(j10) &= \pi - \frac{\pi}{2} - \pi - \tan^{-1}\left(\frac{10}{-99}\right) \\
&\approx -84^\circ\,.
\end{aligned} \tag{5}
$$

The system with $G_2(s)$ as transfer function has no steady-state response since it is unstable **(1 pt)**.

(d) The correct Bode plot is (a) **(1 pt)**. The first step is to rewrite the transfer function in standard-form:

$$
G_2(s) = \frac{10}{\left(\frac{s}{10}+1\right)\left(\frac{s}{-10}+1\right)(s^2+s+1)}\,. \tag{6}
$$

The Bode gain is indeed 20 dB and then there are two complex conjugate poles at $\omega = 1$ and two poles at $\omega = 10$, one stable and one unstable. Each pole contributes with a -20 dB/dec on the magnitude **(1 pt)**. The complex conjugate poles lead to a -180° contribution in terms of phase while the phase contributions of the stable and unstable poles at $\omega = 10$ cancel out **(1 pt)**. Yes, there is resonance since $\zeta = 0.5$ **(1 pt)**.

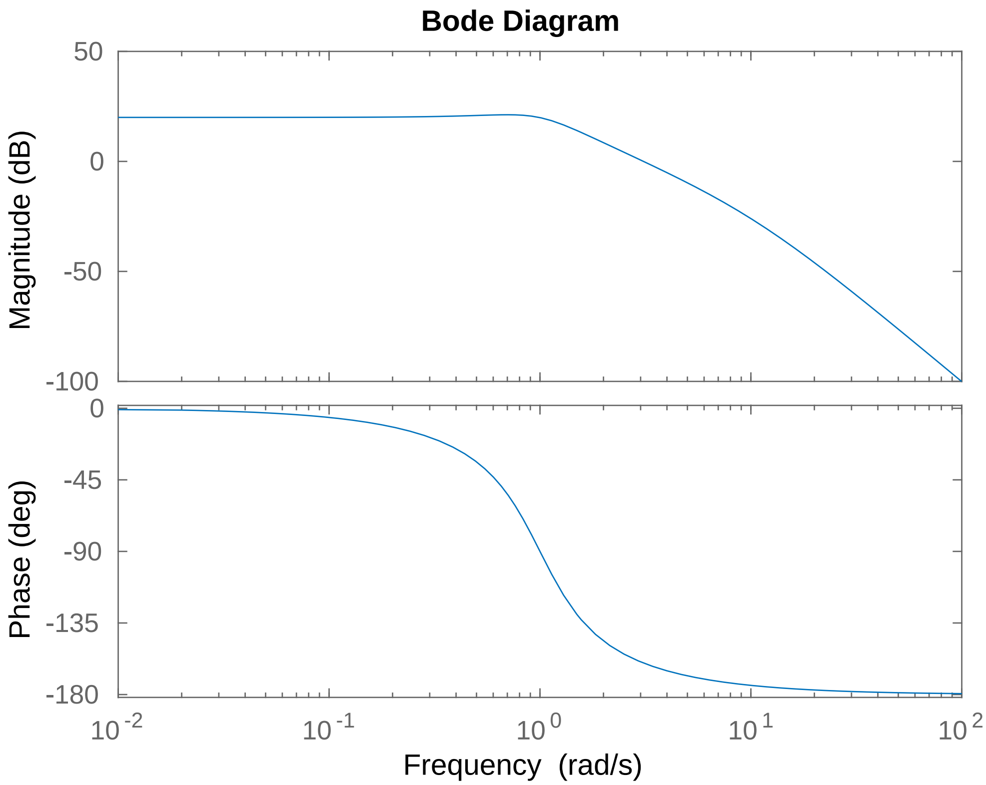


Figure 1: Bode plot of $G_2(s)$.

2. (a) We compute the transfer function from the following equation

$$G(s) = C(sI - A)^{-1}B + D\,. \tag{7}$$

$$(sI - A)^{-1} = \frac{1}{s(s+2)}\begin{bmatrix} s+1 & -1 \\ -1 & s+1 \end{bmatrix} \quad \textbf{(1 pt)}\,. \tag{8}$$

$$\begin{aligned} G(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix}(sI - A)^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \\ &= \frac{s+1}{s(s+2)} - \frac{1}{s(s+2)} \\ &= \frac{1}{s+2} \quad \textbf{(1 pt)}\,. \end{aligned} \tag{9}$$

(b) The transfer function has one pole at $p_1 = -2$ while the system is of second oder. In this scenario, the eigenvalues of the system are not corresponding to the poles of the transfer function because there is a zero-pole cancellation in zero and therefore it is not possible to conclude that the system is marginally stable only via inspection of the poles of its transfer function **(2 pt)**.

(c) As mentioned in the previous point, the eigenvalues of the system are not corresponding to the poles of the transfer function. This is due to a cancellation which corresponds to a non-observable and/or non-controllable mode of the system **(2 pt)**.

Exercise 4

1	2	3	4	Exercise
3	8	9	5	25 Points

1. Equilibrium points are obtained by setting system's dynamics to zero **(1 p.)**. Hence, we have $\frac{d}{dt}x_2 = 0$, yielding $\hat{x}_1 = 0$ **(1 p.)**. Combining it with $\frac{d}{dt}x_1 = 0$ we have $\hat{x}_2 = 0$ **(1 p.)**.

2. The derivative of Lyapunov function is given by

$$\begin{aligned}\dot{V}(x) = \frac{d}{dt}V(x) &= \frac{\partial}{\partial x_1}V(x)\frac{d}{dt}x_1(t) + \frac{\partial}{\partial x_2}V(x)\frac{d}{dt}x_2(t), \textbf{ (1 p.)}\\ &= 2x_1\frac{d}{dt}x_1(t) + 2x_2\frac{d}{dt}x_2(t)\\ &= -8x_1^2(1-x_1^2) \textbf{ (1 p.)}\end{aligned}$$

On the given open set S we have

(a) $V(\hat{x}) = 0$ **(1 p.)**

(b) $V(x) > 0$ for all $x \in S \setminus \hat{x}$ because $V(x)$ is quadratic **(1 p.)**

(c) $\dot{V}(x) \leq 0$ when $x \in S$, since $x_1^2 < 1$ **(1 p.)**.

Hence, according to the Lyapunov direct method, equilibrium $\hat{x}$ is stable **(1 p.)**.

No, the asymptotic stability cannot be concluded via the direct Lyapunov method **(1 p.)** because $\dot{V}(x) = 0$ for $x_1 = 0$, $x_2^2 < 1$, and $x_2 \neq 0$ **(1 p.)**. Hence $\frac{d}{dt}V(x) < 0$ for all $x \in S \setminus \{\hat{x}\}$ does not hold.

3. We recall that $\frac{d}{dt}V(x) = -8x_1^2(1-x_1^2)$. When $x \in \Omega$, $\frac{d}{dt}V(x) \leq 0$ always holds. In addition, Ω is compact, hence it is invariant **(1 p.)**.

The set $\bar{\Omega} = \{x \in \Omega \mid \nabla V(x)f(x) = 0\} = \{x \in \Omega \mid (0, x_2),\ (\pm 1, x_2)\}$ **(1 p.)**. Essentially, the set $\overline{\Omega}$ is consist of three vertical lines cutting the x_1 axis at $\{-1, 0, 1\}$. If $x_1 = 0$ and $x_2 \neq 0$, $\frac{d}{dt}x_1 \neq 0$, which implies we leave the vertical line at 0, either to the left $x_2 > 0$ or to the right $x_2 < 0$ **(1 p.)**. If $x_1 = 1$ and $x_2 \neq 0$, we instantaneously leave the vertical line cutting the x_1 axis at 1 **(1 p.)**. However, if $x_1 = 1$ and $x_2 = 0$, $\dot{x}_1 = 0$, so the previous argument does not hold. Here we notice that $\frac{d}{dt}x_2 \neq 0$, so soon $x_2 \neq 0$, and we are back to the previous case **(1 p.)**. Similar argument for $x_1 = -1$ **(1 p.)**. Finally we can conclude that the maximal invariant set inside $\bar{\Omega}$ is $M = \{\hat{x}\}$ (when $x = \hat{x}$ the system stays at that point); according to the LaSalle theorem $\hat{x}$ locally asymptotically stable **(1 p.)**.

The whole level set $\mathcal{L}(1) = \Omega$ is a region of attraction **(1 p.)**. Since $\frac{d}{dt}V(x) = -8x_1^2(1-x_1^2)$, we cannot find a constant $\ell > 1$ for which $\frac{d}{dt}V(x) \leq 0$ for all $x \in \mathcal{L}(\ell)$; e.g., the point $(l, 0)$ violates this condition **(1 p.)**. Hence, the set Ω is maximal region of attraction that we can estimate with the level sets of $V(x)$.

4. The set Ω is not the maximal region of attraction, because the solution $x(t)$ with initial state $x(0) \in \{(1.02, 1.75), (-0.5, 1.75), (-1, 2), (-0.95, 0.5)\}$ converges to the equilibrium, but these initial states are not in Ω, since $V(x(0)) > 1$. (**(2 p.)** for identifying states that are outside Ω, **(3 p.)** for complete explanation why Ω is not the maximal region of attraction)