Automatic Control Laboratory
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D-ITET
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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	Exercise
5	5	4	5	6	25 Points

1. It follows from

$$y(t) = \alpha h_2(t), \qquad f(t) = \beta(h_1(t) - h_2(t)),$$

$$[\textbf{1 point}]\ \dot{h}_1(t) = \frac{1}{v_1}(u_1(t) - f(t)) \quad \text{and} \quad [\textbf{1 point}]\ \dot{h}_2(t) = \frac{1}{v_2}(u_2(t) + f(t) - y(t)),$$

we have

$$[\textbf{2 + 1 points}]\ \dot{h}(t) = \begin{bmatrix} -\frac{1}{v_1}\beta & \frac{1}{v_1}\beta \\ \frac{1}{v_2}\beta & \frac{-1}{v_2}(\alpha+\beta) \end{bmatrix} \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{v_1} & 0 \\ 0 & \frac{1}{v_2} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$y(t) = [0 \quad \alpha] \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}.$$

2. $A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

Stability: [**2 points**]

$$\begin{aligned} \det(sI - A) &= (1+s)(3-s) + 1 \\ &= s^2 + 3s + 1 \end{aligned}$$

Then, we have

$$s_1 = \frac{-3+\sqrt{5}}{2} \quad \text{and} \quad s_2 = \frac{-3-\sqrt{5}}{2}.$$

All $\text{Re}[s_i] \leq 0$, i.e., the system is stable.

Controllability: [**1 point**] P:= $\begin{bmatrix} B & AB \end{bmatrix}$. Since P has full row rank, and therefore the system is controllable.

Observability: [**2 points**] O:= $\begin{bmatrix} C \\ CA \end{bmatrix}$. Since O has full column rank, and therefore the system is observable.

3. The transfer function is defined as [**2 points**] $G(s) = C(sI - A)^{-1}B$, therefore, a direct calculation yields

$$[\textbf{2 points}] \quad G(s) = \frac{1}{s^2 + 3s + 1}[1 \ s+1].$$

4. Remember that from final value theorem, we have

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)U(s) = \lim_{s\to 0} sG(s)[\frac{1}{s}\ \frac{1}{s}]^T = G(0)[1\ 1]^T = 2.$$

   Each equality accounts for [**1 point**].

5. The singular values for constant input signals are given by the square root of the largest and smallest eigenvalues of the matrix $G(0)^T G(0)$. The eigenvalues of $G(0)^T G(0)$ are 0 and 2.

   [**2 points**] The largest singular value corresponds to the input signal vector

$$u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

   [**1 point**] which gives the steady state output signal $y(t) = 2$.

   [**2 points**] The smallest singular value corresponds to the input signal vector

$$u(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

   [**1 point**] which gives the steady state output signal $y(t) = 0$.

   Alternatively, it follows from final value theorem that

$$[\textbf{2 points}] \quad \lim_{t\to\infty} y(t) = \lim_{s\to 0} G(s)u(t) = [1 \quad 1] \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = u_1(t) + u_2(t),$$

   which implies that the steady state output is $u_1(t) + u_2(t)$. Since $\|u(t)\| = 1$, [**2 points**] the absolute value of the sum $|u_1(t) + u_2(t)|$ is maximized if $u_1(t) = u_2(t) = \frac{1}{\sqrt{2}}$, while [**2 point**] the absolute value of the sum $u_1(t) + u_2(t)$ is minimized if $u_1(t) = -u_2(t) = \frac{1}{\sqrt{2}}$.

Exercise 2

1	2	3	4	5	Exercise
3	6	3	7	6	25 Points

1. Noticing that the matrix A is upper triangular, its eigenvalues are $\lambda_1 = -2, \lambda_2 = -3$ and $\lambda_3 = a$. **2 p.** So the system will be asymptotically stable for $a < 0$. **1 p.**

2. The controllability matrix is

$$P = [B\ AB\ A^2B] = \begin{bmatrix} 0 & a & -5a-1 \\ 0 & 1 & a-3 \\ 1 & -3 & 9 \end{bmatrix}.\mathbf{1\ p.}$$

The submatrix $\begin{bmatrix} a & -5a-1 \\ 1 & a-3 \end{bmatrix}$ has determinant $(a+1)^2$ **1 p.**. Thus, if $a = -1$, the rank of P is not 3, and the system is uncontrollable. **1 p.**

For $a = -1$, since $\text{range}\{P\} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ **1 p.**, the reachable space is the span of these vectors, so from 0 the only reachable states are those where the first component is equal and opposite in sign to the second **1 p.**

For $a \neq -1$, the matrix P is full rank. Hence, the system is controllable for $a \neq -1$ and the state $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is reachable. **1 p.**

3. The observability matrix is

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & a & -2 \\ 0 & a^2 & a+6 \end{bmatrix}.\mathbf{1\ p.}$$

which will never have full rank for any a. Thus the system will never be observable. **2 p.**

4. The system is unobservable for all values of a. Hence, there will be at least one pole-zero cancellation in the transfer function, i.e., the transfer function will have less than three poles. **1 p.**
To determine whether the transfer function has one or two pole-zero cancellations, we have to determine for which values of λ the matrices

$$\begin{bmatrix} C \\ \lambda I - A \end{bmatrix} \text{ and } [B\ \lambda I - A]$$

lose rank.

It is easy to see that

$$\begin{bmatrix} C \\ \lambda I - A \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ \lambda+2 & 1 & -a \\ 0 & \lambda - a & -1 \\ 0 & 0 & \lambda+3 \end{bmatrix} \textbf{1 p.}$$

loses rank for $\lambda = -2$. **1 p.**

On the other hand, the matrix

$$[B \ \lambda I - A] = \begin{bmatrix} 0 & \lambda+2 & 1 & -a \\ 0 & 0 & \lambda - a & -1 \\ 1 & 0 & 0 & \lambda+3 \end{bmatrix} \textbf{1 p.}$$

evaluated at $a = -1$ is

$$\begin{bmatrix} 0 & \lambda+2 & 1 & 1 \\ 0 & 0 & \lambda+1 & -1 \\ 1 & 0 & 0 & \lambda+3 \end{bmatrix} \textbf{1 p.}$$

which loses rank for $\lambda = -2$. **1 p.**

Therefore, we conclude that for any a, only the pole $\lambda = -2$ is cancelled, and the other two will appear in the transfer function. **1 p.**

5. This means that the original system is stable. **1 p.**

   Considering Q, the square root of P, such that $Q^2 = P$, we have

$$Q^2 A + A^T Q^2 = -I, \textbf{1 p.}$$

   multiplying both sides by Q^{-1} we have

$$QAQ^{-1} + Q^{-1}A^T Q = -P^{-1}. \textbf{1 p.}$$

   So considering the transformation $T = Q$ **1 p.**, we have $\hat{A} = QAQ^{-1}$ **1 p.** and thus

$$\hat{A} + \hat{A}^T = -P^{-1}. \textbf{1 p.}$$

Exercise 3

1	2	3	4	5	Exercise
4	5	8	5	3	25 Points

1. The transfer function is given by $G(s) = C(sI - A)^{-1}B$ **(1 pt.)**, for the system Σ_1 this results in

$$G_1(s) = \begin{bmatrix} 0 & \frac{1000}{2} \end{bmatrix} \begin{bmatrix} s+12 & 11 \\ -2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \textbf{(1 pts.)}$$
$$= \begin{bmatrix} 0 & \frac{1000}{2} \end{bmatrix} \frac{1}{(s+1)(s+10)} \begin{bmatrix} s-1 & -11 \\ 2 & s+12 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \textbf{(1 pts.)}$$
$$= \frac{1000}{(s+1)(s+10)}, \textbf{(1 pts.)}.$$

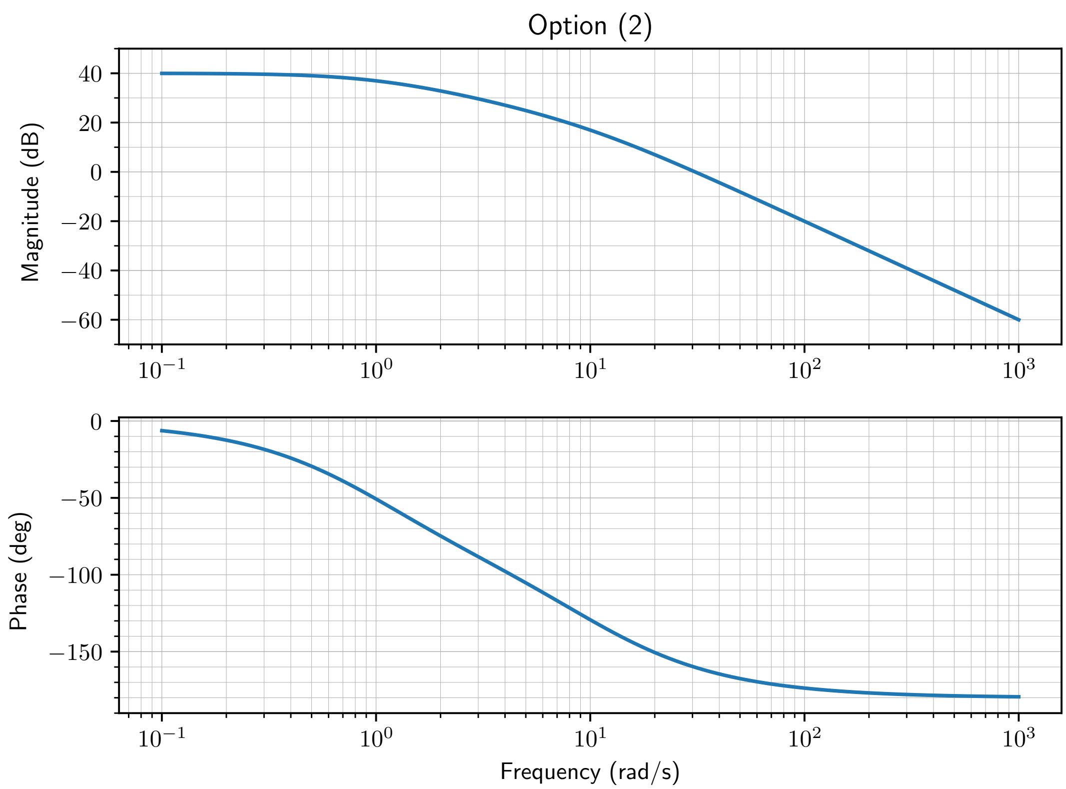

Figure 1: Bode plot for Σ_1

2. Option 2, see Figure 1, **(1 pt).** each for starting value, changes due to each pole in both magnitude and phase plots. ⇒ **(5 pts.)**

(OR)

Option 1 has incorrect phase plot because it drops only by 60° per pole **(1 pt.)**.

Option 3 has an incorrect gain **(1 pt.)**.

Option 4 has an incorrect pole **(1 pt.)**.

So Option 2 is the correct answer **(2 pt.)**.

3. Using the fact that Σ_1 is asymptotically stable and its phase and magnitude are monotonically decreasing we see that one can use the Bode stability criterion **(2 pt.)**.

   The phase margin is infinite, since the magnitude is below 1 for all ω **(2 pt.)**.

   The phase plot attains the value -180° roughly at $\omega \approx 100^c/s$ **(1 pt.)** and $|G_f(100j)| \approx 10^{-4}$ **(1 pt.)**, so the gain margin is approx. 10^4 **(1 pt.)**, the closed loop system will be stable for $K < 10^4$ **(1 pt.)**.

4. We want $Z = 0$. Since we have $P = 0$, we want $N = 0$ **(1 pt.)**. Therefore, only those K for which the point $(-1/K, 0)$ has 0 encirclements is stable **(1 pt.)**. This is possible for approx. $K < 10^4$ **(1 pt.)** and $K > -10^1$ **(1 pt.)**. But we only want the positive branch of the solutions **(1 pt.)**.

5. For $K = 2 * 10^4$, we have at $\frac{-1}{K} = -5 * 10^{-5}$ **(1 pt.)** two clockwise encirclements in the nyquist plot, so $N = 2$ **(1 pt.)**. So we will have $Z = N + P = 2$ unstable poles in the closed loop system **(1 pt.)**.

Exercise 4

1	2	3	4	5	Exercise
3	4	8	7	3	25 Points

1. In order to show that S_0 is invariant, it is sufficient to analyse the vector field at the boundaries of the positive octant. The vector field evaluated at $x_2 = 0$ is $\dot{x}_1 = \alpha - \mu x_1$, $\dot{x}_2 = 0$, so the trajectories cannot escape the positive quadrant from that side [**1.5pts**]. Moreover, when $x_1 = 0$, we have that $\dot{x}_1 = \alpha > 0$ and $\dot{x}_2 = -(\gamma + \mu)x_2$ [**1.5pts**]. Hence, S_0 is an invariant set.

2. As usual, the equilibria are found by imposing [**1pt**]

$$\alpha - \mu x_1 - \beta x_1 x_2 = 0,$$
$$(\beta x_1 - (\gamma + \mu))x_2 = 0.$$

   We have two isolated equilibrium points. In fact, the second equation is true for either

   - $x_2 = 0$; in this case, from the first equation, $x_1 = \alpha/\mu$ [**1pt**]; or,
   - $\beta x_1 - (\gamma + \mu) = 0$; so we have $x_1 = (\gamma + \mu)/\beta$ and, substituting in the first equation, $x_2 = \frac{\alpha\beta - \mu(\gamma+\mu)}{\beta(\gamma+\mu)} = \frac{\mu}{\beta}(R_0 - 1)$ [**2pts**].

   The two equilibrium points are then

$$x_{DFE} = \begin{bmatrix} \frac{\alpha}{\mu} \\ 0 \end{bmatrix} \quad \text{and} \quad x_{EE} = \begin{bmatrix} \frac{\gamma+\mu}{\beta} \\ \frac{\mu}{\beta}(R_0 - 1) \end{bmatrix},$$

   and are usually called the *disease-free equilibrium* (DFE) and the *endemic equilibrium* (EE), respectively.

3. First, we compute the Jacobian matrix [**1pt**] as

$$J = \begin{bmatrix} -\beta x_2 - \mu & -\beta x_1 \\ \beta x_2 & \beta x_1 - \gamma - \mu \end{bmatrix}.$$

   - DFE [**3pts**]: the Jacobian evaluated at the DFE is

$$J_{DFE} = \begin{bmatrix} -\mu & -\frac{\alpha\beta}{\mu} \\ 0 & \frac{\alpha\beta}{\mu} - \gamma - \mu \end{bmatrix},$$

     and, since it is an upper triangular matrix, its eigenvalues are $\lambda_1 = -\mu$ and $\lambda_2 = \frac{\alpha\beta}{\mu} - (\gamma + \mu)$. Hence, the EE is locally asymptotically stable for $R_0 < 1$ and unstable for $R_0 > 1$. If $R_0 = 1$, one eigenvalue is zero and nothing can be concluded.

- EE **[3pts]**: the Jacobian in this case is

$$J_{EE} = \begin{bmatrix} -\mu R_0 & -(\gamma+\mu) \\ \mu(R_0 - 1) & 0 \end{bmatrix},$$

and, after some algebra, its characteristic polynomial is

$$\chi(\lambda) = \lambda^2 + \mu R_0 \lambda + \alpha\beta - \mu(\gamma+\mu).$$

Thanks to the Hurwitz criterion for the coefficients of a polynomial, we can conclude that the EE is unstable for $R_0 < 1$ and locally asymptotically stable for $R_0 > 1$ (i.e. the opposite of the DFE). Again, if $R_0 = 1$, one eigenvalue is zero and nothing can be concluded.

The basic reproduction number determines whether the virus will go extinct or will remain permanently endemic in the population **[1pt]**.

4. When $R_0 = 1$ the EE coincides with the DFE, so the only equilibrium point is $x^* = (\frac{\alpha}{\mu}, 0)$.

(a) Let's compute the Lie derivative of V along system's trajectories **[2pts]**:

$$\begin{aligned} \frac{d}{dt}V(x(t)) = \nabla V f &= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{x_1 - x_1^*}{x_1} & 1 \end{bmatrix} \begin{bmatrix} \alpha - \mu x_1 - \beta x_1 x_2 \\ \beta x_1 x_2 - (\gamma+\mu)x_2 \end{bmatrix} \\ &= -\frac{\frac{1}{\mu}(\mu x_1 - \alpha)^2}{x_1}. \end{aligned} \tag{1}$$

In particular, when $x(t) \in S_0$ it holds $\nabla V f \le 0$. Hence $\nabla V f \le 0$ in S_K as well because $S_K \subseteq S_0$. As a consequence, S_K is an invariant set for our system **[2pts]**.

(b) By virtue of LaSalle's theorem, all the trajectories starting in S_K will converge to the largest invariant set in $M = \{x \in S_K \,|\, \nabla V f = 0\} = \{x \in S_K \,|\, x_1 = \alpha/\mu\}$ **[1pts]**. Since the vector field evaluated in M is $\dot{x}_1 = -\beta(\alpha/\mu)x_2$, $\dot{x}_2 = 0$ (i.e. $\dot{x} = 0$ in S_K iff $x_2 = 0$), we can conclude that x^* is the largest invariant set in M and hence all the trajectories starting in S_K converge to x^* **[2pts]**.

5. In this case, we have that the DFE is asymptotically stable when $R_0(1-p) < 1$ **[2pts]**. In the case of COVID-19, the minimum vaccination rate to eradicate the virus is **[1pt]**

$$p > 1 - \frac{1}{R_0} = \frac{2}{3}.$$

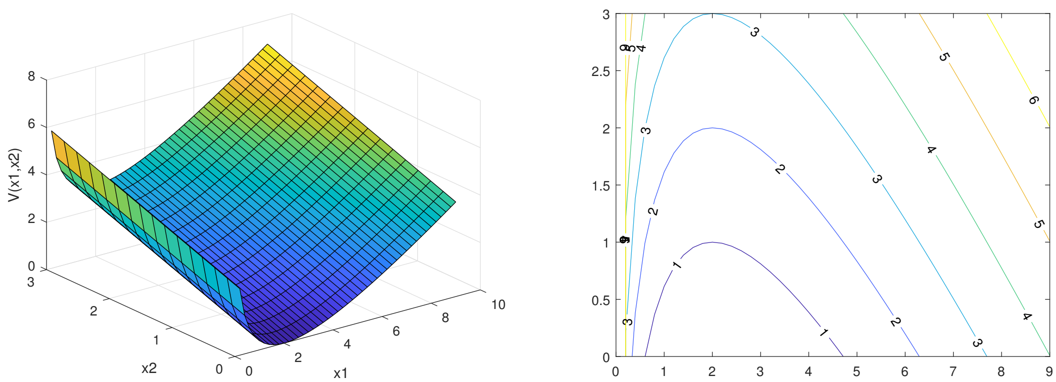

Figure 2: The Lyapunov function $V(x)$ is depicted together with its level curves in the positive octant. In this specific plot we fixed $\alpha/\mu = 2$.