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# Signals and Systems II
# 4. Semester, BSc

# Solutions

# Exercise 1

| 1 | 2 | 3 | Exercise |
|---|---|---|---|
| 10 | 11 | 6 | 27 Points |

1. The Buck converter with the additional switch $S'$, is represented in Figure 1. Note that $S'$ is closed when $S$ is open, and vice versa.

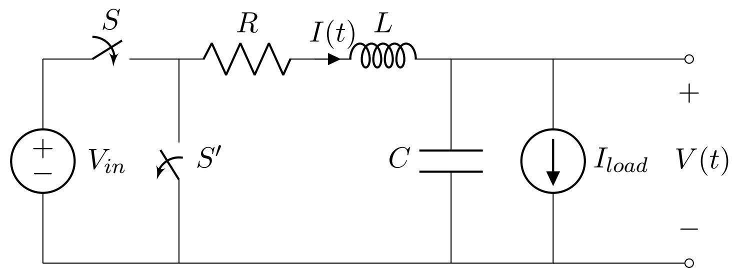



Figure 1: Buck converter electrical model

(a) When the switch $S$ is **closed**, the equations representing the circuit are

$$\begin{aligned} KCL: &\quad I(t) - I_{load} - C\frac{dV(t)}{dt} = 0 \\ KVL: &\quad V_{in} - RI(t) - L\frac{dI(t)}{dt} - V(t) = 0. \end{aligned}$$

If we define $x(t) = \begin{bmatrix} V(t) & I(t) \end{bmatrix}'$, we can re-write them in the following state-space form

$$\dot{x}(t) = f_c(x(t)) = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} x(t) + \begin{bmatrix} -I_{load}/C \\ V_{in}/L \end{bmatrix}. \quad \textbf{[4P]}$$

On the other hand, when the switch $S$ is **open**, we have

$$\begin{aligned} KCL: &\quad I(t) - I_{load} - C\frac{dV(t)}{dt} = 0 \\ KVL: &\quad -RI(t) - L\frac{dI(t)}{dt} - V(t) = 0, \end{aligned}$$

that leads to

$$\dot{x}(t) = f_o(x(t)) = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} x(t) + \begin{bmatrix} -I_{load}/C \\ 0 \end{bmatrix}. \quad \textbf{[4P]}$$

(b) By applying the averaging operation to the two systems, we obtain the following average dynamics

$$\dot{x}(t) = f_{avg}(x(t)) = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} x(t) + \begin{bmatrix} -I_{load}/C \\ uV_{in}/L \end{bmatrix}. \quad \textbf{[2P]}$$

2. (a) Since $u(t)$ is the control input, we can re-write the system in the following form

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ V_{in}/L \end{bmatrix}}_{B} u(t). \quad \textbf{[1P]} \tag{1}$$

Let's now inspect the stability through the matrix $A$. Its characteristic polynomial is

$$\det(\lambda\mathcal{I} - A) = \begin{vmatrix} \lambda & -1/C \\ 1/L & \lambda + R/L \end{vmatrix} = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC},$$

which has roots

$$\lambda_{1,2} = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}.$$

For positive electrical parameters, we have that $\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} < \frac{R}{L}$. Therefore, $\mathrm{Re}(\lambda_i) < 0$ for both $i = 1, 2$, and we can conclude that the system is asymptotically stable **[3P]**.

Note that, as an alternative, we could have exploited the Hurwitz criterion. In fact, asymptotic stability is derived from the fact that the three coefficients of the characteristic polynomial ($1$, $R/L$, $1/LC$) have the same sign.

(b) The equilibrium condition is obtained by imposing $\dot{x} = 0$, and it leads to

$$x = \begin{bmatrix} V_{in}\bar{u} \\ 0 \end{bmatrix}. \quad \textbf{[2P]}$$

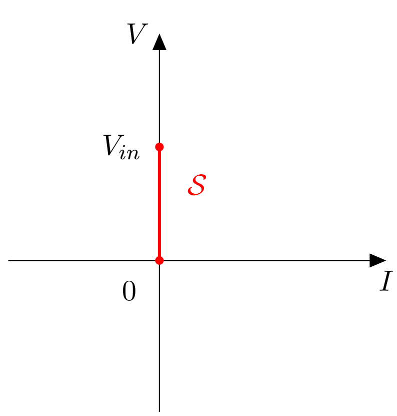
The first equation is $V = V_{in}\bar{u}$. By dividing both sides by $V_{in}$ we obtain

$$V/V_{in} = \bar{u} \in [0, 1],$$

and therefore the condition $V/V_{in} \leq 1$ holds **[1P]**.

(c) From the previously computed equilibrium condition it is evident that, with a constant input law $\bar{u} \in [0, 1]$, it is possible to stabilize the system only in the following subset of the state-space

$$\mathcal{S} = \left\{ x \mid x \in \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} V_{in} \\ 0 \end{bmatrix} \right) \right\} \quad \textbf{[2P]}$$




**[2P]**

3. (a) Let's call $\bar{p} = \begin{bmatrix} 1 & 1 \end{bmatrix}'$. Since $\bar{p} \notin \mathcal{S}$, it is not possible to find an admissible $\bar{u}$ such that $\lim_{t\to\infty} x(t) = \bar{p}$ **[2P]**.

(b) If $u$ was unconstrained, the previously computed set $\mathcal{S}$ would be

$$\mathcal{S} = \left\{ x \mid x \in \left( \begin{bmatrix} -\infty \\ 0 \end{bmatrix}, \begin{bmatrix} +\infty \\ 0 \end{bmatrix} \right) \right\},$$

that corresponds to the vertical line $I = 0$. Therefore, any equilibrium with $I \neq 0$ is not compatible with our system, including $\bar{p}$ **[2P]**.

(c) The previous conditions are not in contrast with the outcome of the stability test. In fact, the asymptotic stability property of the system only guarantees that the trajectory will converge to the equilibrium. What is not guaranteed is the ability to stabilize the system in any arbitrary point of the state-space. We know that the trajectory will converge to one of the points in $\mathcal{S}$, depending on which particular $\bar{u}$ is selected **[2P]**.

# Exercise 2

| 1 | 2 | 3 | 4 | Exercise |
|---|---|---|---|---|
| 4 | 5 | 6 | 8 | 23 Points |

1. (**4P** in total)

   (a) The characteristic polynomial is

   $$\det(sI - A_1) = s^2 + 3s - 4 = (s-1)(s+4),$$

   which implies that the poles of $A_1$ are 1, -4. The system S1 has a RHP pole and it is unstable. [**2P**]

   (b) The controllability matrix is given by

   $$W_c = \begin{bmatrix} B_1 & A_1 B_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

   Obviously, it has full rank. The system S1 is controllable. [**1P**]

   (c) The observablity matrix is given by

   $$W_o = \begin{bmatrix} C_1 \\ C_1 A_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$$

   It also has full rank. Therefore, the system S1 is observable. [**1P**]

2. (**5P** in total)

   (a) By Figure 1, we have the following equations

   $$\begin{aligned} sX_2(s) &= U_2(s) - 2X_2(s), \\ Y(s) &= 2X_2(s). \quad [\mathbf{1P}] \end{aligned}$$

   Hence, the transfer functions of S2 are

   $$\begin{aligned} \frac{X_2(s)}{U_2(s)} &= \frac{1}{s+2}, \quad [\mathbf{1P}] \\ \frac{Y(s)}{U_2(s)} &= \frac{2}{s+2}. \quad [\mathbf{1P}] \end{aligned}$$

   (b) By taking the inverse Laplace transform, we get the state space equations of S2:

   $$\begin{aligned} \dot{x}_2(t) &= -2x_2(t) + u_2(t), \quad [\mathbf{1P}] \\ y(t) &= 2x_2(t). \quad [\mathbf{1P}] \end{aligned}$$

3. (**6P** in total)

   By Figure 2, we have $u_2(t) = y_1(t) = x_{11}(t) + x_{12}(t)$, which makes:

   $$\dot{x}_2(t) = x_{11}(t) + x_{12}(t) - 2x_2(t), \quad [\mathbf{1P}]$$

so together with $u(t) = u_1(t)$, we have the state space equation of S:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 4 & -3 & 0 \\ 1 & 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}, \quad D = 0. \quad [\mathbf{5P}]$$

(For $A$, it values [**2P**], and $B$, $C$ and $D$ value [**1P**] resp.)

4. (**8P** in total)

   (a) By (1.a), the poles of S1 are $s_1 = 1, s_2 = -4$. By (2a), the pole of S2 is $s = -2$. As S is a cascade system of S1 and S2, the poles of S are $s_1 = 1, s_2 = -4, s_3 = -2$ [**1P**]. There exists one RHP pole and S is unstable [**1P**].

   (b) Under zero input, the zero input transition $x(t) = e^{At}x(0)$. By the hint, we have

   $$x(t) = \begin{bmatrix} (\frac{4a}{5} + \frac{b}{5})e^t + (\frac{a}{5} - \frac{b}{5})e^{-4t} \\ (\frac{4a}{5} + \frac{b}{5})e^t + (\frac{-4a}{5} + \frac{4b}{5})e^{-4t} \\ (\frac{8a}{15} + \frac{2b}{15})e^t + (\frac{3a}{10} - \frac{3b}{10})e^{-4t} - (\frac{5a}{6} - \frac{b}{6} - c)e^{-2t} \end{bmatrix}, \quad [\mathbf{1P}]$$

   Since there exists an unstable pole $s_1 = 1$, we need to make the coefficient of the term $e^t$ in $x(t)$ to be zero, in order to keep $x$ finite as $t \to \infty$. [**2P**] Then we have $b = -4a$. [**1P**] And $x(\infty) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

   (c) There is no contradiction. When a system is unstable, the output of the system may be infinite even though the input to the system was finite. It doesn't mean that for every initial state, the output of the system is infinite. In (4b), the input is zero, and the output of the system could be infinite if $b \neq -4a$, which also indicates that S is unstable. [**2P**]

# Exercise 3

| 1 | 2 | 3 | Exercise |
|---|---|---|---|
| 8 | 7 | 10 | 25 Points |

1. (a) For an equilibrium point we should have:

$$\begin{array}{l} x_1 = \frac{\alpha}{1+x_1^2+x_2^2} x_2 \\ x_2 = \frac{\alpha}{1+x_1^2+x_2^2} x_1 \end{array}, \textbf{[2P]}$$

It can clearly be observed that $(0,0)$ is an equilibrium point for any $\alpha$. **[1P]**
By concatenating both equations we get $x_1 = \frac{\alpha^2}{(1+x_1^2+x_2^2)^2} x_1$. So at any non-trivial equilibrium we need to have $\frac{\alpha^2}{(1+x_1^2+x_2^2)^2} = 1$ **[1P*]**.
Since $\alpha > 0$ and $(1 + x_1^2 + x_2^2) > 0$, then we need to have $\alpha = (1 + x_1^2 + x_2^2)$. This equation has a solution, because $\alpha \geq 1$. **[1P*]**
Substituting this equation $\alpha = (1+x_1^2+x_2^2)$ in one of the equilibrium equations we get $x_1 = \frac{\alpha}{1+x_1^2+x_2^2} x_2 = x_2$. **[1P*]**
Then at equilibrium we need to have either $x_1 = x_2$ and $\alpha = (1 + x_1^2 + x_2^2) = (1 + x_1^2 + x_1^2)$, or $x_1 = x_2 = 0$. So the equilibrium points are $x_1 = x_2 = \pm\sqrt{\frac{\alpha-1}{2}}$ and $x_1 = x_2 = 0$. **[1P]**
Alternatively, instead of the points marked with *, it's also correct to first prove $x_1^2 = x_2^2$, then $x_1 = x_2$, and then $\alpha = (1 + x_1^2 + x_2^2)$.

   (b) For $\alpha < 1$, the equation $\alpha = (1 + x_1^2 + x_2^2)$ has no solution. So the only equilibrium point in $(0,0)$. **[1P]**

2. To linearize the system we need the derivatives of $x_1(k+1)(x_1(k), x_2(k))$ and $x_2(k+1)(x_1(k), x_2(k))$ around the equilibrium point $(0,0)$:

$$\begin{array}{l} \frac{\partial x_1(k+1)}{\partial x_1(k)}|_{(0,0)} = \alpha(-2x_2(k)x_1(k)(1 + x_1(k)^2 + x_2(k)^2)^{-2})|_{(0,0)} = 0 \\ \frac{\partial x_1(k+1)}{\partial x_2(k)}|_{(0,0)} = \alpha((1 + x_1(k)^2 + x_2(k)^2)^{-1} - 2x_2(k)x_2(k)(1 + x_1(k)^2 + x_2(k)^2)^{-2})|_{(0,0)} = \alpha \\ \frac{\partial x_2(k+1)}{\partial x_1(k)}|_{(0,0)} = \alpha((1 + x_1(k)^2 + x_2(k)^2)^{-1} - 2x_1(k)x_1(k)(1 + x_1(k)^2 + x_2(k)^2)^{-2})|_{(0,0)} = \alpha \\ \frac{\partial x_2(k+1)}{\partial x_2(k)}|_{(0,0)} = \alpha(-2x_2(k)x_1(k)(1 + x_1(k)^2 + x_2(k)^2)^{-2})|_{(0,0)} = 0, \textbf{[2P]} \end{array}$$

Now we can linearize the nonlinear discrete-time equation around $(0,0)$:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \textbf{[1P]}$$

which has eigenvalues $\begin{vmatrix} \lambda & -\alpha \\ -\alpha & \lambda \end{vmatrix} = \lambda^2 - \alpha^2 = 0$ **[1P, give anyway if step is omitted, but next is correct]** so $\lambda = \alpha, -\alpha$. **[2P]**
Therefore the equilibrium $(0,0)$ is locally asymptotically stable if $\alpha < 1$. **[1P]**

3. For $\alpha = 1$ we have the system:

$$x_1(k+1) = x_2(k)(1 + x_1(k)^2 + x_2(k)^2)^{-1}$$
$$x_2(k+1) = x_1(k)(1 + x_1(k)^2 + x_2(k)^2)^{-1}$$

We check if the given function $V((x_1(k), x_2(k))) = x_1(k)^2 + x_2(k)^2$ satisfies the conditions of the Lyapunov second method with $\tilde{x} = (0, 0)$:

i. $V((0,0)) = 0^2 + 0^2 = 0$ **[1P]**

ii. $V((x_1, x_2)) = x_1^2 + x_2^2 > 0,\ \forall (x_1, x_2) \neq (0,0)$ **[2P]**

iv. $V((x_1(k+1), x_2(k+1))) = x_1(k+1)^2 + x_2(k+1)^2$
$= (x_1(k)^2 + x_2(k)^2)(1 + x_1(k)^2 + x_2(k)^2)^{-2}$ **[1P]**
$< x_1(k)^2 + x_2(k)^2 = V((x_1(k), x_2(k))),\ \forall (x_1(k), x_2(k)) \neq (0,0),$
because $(1 + x_1(k)^2 + x_2(k)^2) > 1\ \forall (x_1(k), x_2(k)) \neq (0,0)$ **[2P]**

v. The analysis is for $S = \mathbb{R}^2$ (To get full credit, there is no need to write this as a separate bullet as long as the rest are proven for all $x \in \mathbb{R}^2$). **[1P]**

vi. $\|(x_1, x_2)\| \to \infty \implies V((x_1, x_2)) = \|(x_1, x_2)\|^2 \to \infty$ (needs to be justified in some way, for example indicating that $V(x)$ is the square of the norm)**[2P]**

So the the equilibrium point $(0, 0)$ is globally asymptotically stable. **[1P]**

# Exercise 4

| 1 | 2 | 3 | 4 | 5 | Exercise |
|---|---|---|---|---|---|
| 3 | 5 | 4 | 7 | 6 | 25 Points |

1. Setting denominator to 0 we obtain $(s-\sigma)^2 = -\omega_0^2$. Hence, the poles are $p_{1,2} = \sigma \pm j\omega_0$.**[2P]** Thus, the system is asymptotically stable if and only if $\mathrm{Re}\, p_{1,2} < 0$, i.e. $\sigma < 0$. **[1P]**

2. Note that

$$G(s) = \frac{Y(s)}{U(s)}$$
$$(s^2 - 2s\sigma + \sigma^2 + \omega_0^2)Y(s) = \omega_0^2 U(s)$$

which leads to the following ODE, using inverce Laplace transform

$$\ddot{y}(t) - 2\sigma\dot{y}(t) + (\sigma^2+\omega_0^2)y(t) = \omega_0^2 u(t).\textbf{[5P]}$$

3. By putting $x_1(t) = \dot{y}(t)$ and $x_2(t) = y(t)$ we obtain the following system: $A = \begin{bmatrix} 2\sigma & -(\sigma^2+\omega_0^2) \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \omega_0^2 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $D = 0$. **[4P]**

4. For $\sigma = -1$, $\omega_0 = 1$ we have,

(a)

$$G(j\omega) = \frac{1}{-\omega^2 + 2j\omega + 2} = \frac{(2-\omega^2) - 2\omega j}{(2-\omega^2)^2 + 4\omega^2} = \frac{(2-\omega^2) - 2\omega j}{4+\omega^4},$$
$$|G(j\omega)| = \frac{1}{\omega^4+4}\sqrt{(2-\omega^2)^2 + (2\omega)^2} = \frac{1}{\sqrt{\omega^4+4}}, \textbf{[2P]}$$

(b)

$$\angle G(j\omega) = \arctan\left(\frac{-2\omega}{2-\omega^2}\right) = -\arctan(2).\textbf{[2P]}$$

(c) In this case, we can simply evaluate the transfer function at the frequency of the input sine signal to calculate the steady state output signal as

$$y(t) = |G(j\omega)| \sin(\omega t + \angle G(j\omega)).\textbf{[1P]}$$

With $\omega = 1$ **[1P]** we obtain

$$y(t) = |G(j1)| \sin(t + \angle G(1j))$$
$$|G(j1)| = \frac{1}{\sqrt{5}} \text{ (for } \omega = 1)$$
$$\angle G(j\omega) = -\arctan(2) \text{ (for } \omega = 1)$$
$$\rightarrow y(t) = \frac{1}{\sqrt{5}} \sin(t - \arctan(2)).\textbf{[1P]}$$

5. (a) Transfer function of the closed loop is given by $G_1(s) = \frac{1}{1+KG(s)}$**[2P]** $= \frac{(s-\sigma)^2+\omega_0^2}{(s-\sigma)^2+(1+K)\omega_0^2}$.

   (b) Note that $R(s) = \frac{1}{s}$ **[1P]** . For $\sigma = 0$ steady state error, using Final Value Theorem is

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sG_1(s)R(s) = \lim_{s\to 0} s\frac{(s-\sigma)^2+\omega_0^2}{(s-\sigma)^2+(K+1)\omega_0^2}\frac{1}{s} = \frac{1}{K+1}.\textbf{[2P]}$$

   Such $K < \infty$ does not exist, since $\lim_{t\to\infty} e(t) = \frac{1}{K+1}$.**[1P]**.