Automatic Control Laboratory
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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	Exercise
7	3	7	5	3	25 Points

1. We first get the trivial equations for the first and second states:

$$\dot{\theta}(t) = \dot{\theta}(t),$$
$$\dot{\theta}(t) = \dot{\theta}_2(t),$$

which, written in $x(t)$, are

$$\dot{x}_1(t) = x_3(t),$$
$$\dot{x}_2(t) = x_4(t).$$

Solution assuming spring force is given by $k(\theta_1 - \theta_2)$**:**
As for the actual dynamics, we get

$$(m\ell^2)\ddot{\theta}_1(t) = -mg\ell\sin(\theta_1(t)) + k\ell\cos(\theta_1(t))(\theta_2(t) - \theta_1(t)) + T(t)$$
$$(m\ell^2)\ddot{\theta}_2(t) = -mg\ell\sin(\theta_2(t)) + k\ell\cos(\theta_2(t))(\theta_1(t) - \theta_2(t))$$

and hence we finally get

$$\dot{x}(t) = \begin{bmatrix} x_3(t) \\ x_4(t) \\ -\sin(x_1(t)) + \cos(x_1(t))(x_2(t) - x_1(t)) + \frac{u(t)}{m\ell^2} \\ -\sin(x_2(t)) + \cos(x_2(t))(x_1(t) - x_2(t)) \end{bmatrix}$$

Solution assuming spring force is given by $k\ell(\sin\theta_1 - \sin\theta_2)$ **which is physically correct:**
As for the actual dynamics, we get

$$(m\ell^2)\ddot{\theta}_1(t) = -mg\ell\sin(\theta_1(t)) + k\ell^2\cos(\theta_1(t))(\sin(\theta_2(t)) - \sin(\theta_1(t))) + T(t)$$
$$(m\ell^2)\ddot{\theta}_2(t) = -mg\ell\sin(\theta_2(t)) + k\ell^2\cos(\theta_2(t))(\sin(\theta_1(t)) - \sin(\theta_2(t)))$$

and hence we finally get

$$\dot{x}(t) = \begin{bmatrix} x_3(t) \\ x_4(t) \\ -\sin(x_1(t)) + \ell\cos(x_1(t))(\sin(x_2(t)) - \sin(x_1(t))) + \frac{u(t)}{m\ell^2} \\ -\sin(x_2(t)) + \ell\cos(x_2(t))(\sin(x_1(t)) - \sin(x_2(t))) \end{bmatrix}$$

Solution assuming spring force is given by $k\ell(\theta_1 - \theta_2)$**:**
As for the actual dynamics, we get

$$(m\ell^2)\ddot{\theta}_1(t) = -mg\ell\sin(\theta_1(t)) + k\ell^2\cos(\theta_1(t))(\theta_2(t) - \theta_1(t)) + T(t)$$
$$(m\ell^2)\ddot{\theta}_2(t) = -mg\ell\sin(\theta_2(t)) + k\ell^2\cos(\theta_2(t))(\theta_1(t) - \theta_2(t))$$

and hence we finally get

$$\dot{x}(t) = \begin{bmatrix} x_3(t) \\ x_4(t) \\ -\sin(x_1(t)) + \ell\cos(x_1(t))(x_2(t) - x_1(t)) + \frac{u(t)}{m\ell^2} \\ -\sin(x_2(t)) + \ell\cos(x_2(t))(x_1(t) - x_2(t)) \end{bmatrix}$$

Solution assuming spring force is given by $k(\sin\theta_1 - \sin\theta_2)$**:**
As for the actual dynamics, we get

$$\begin{aligned}(m\ell^2)\ddot{\theta}_1(t) &= -mg\ell\sin(\theta_1(t)) + k\ell\cos(\theta_1(t))(\sin(\theta_2(t)) - \sin(\theta_1(t))) + T(t) \\ (m\ell^2)\ddot{\theta}_2(t) &= -mg\ell\sin(\theta_2(t)) + k\ell\cos(\theta_2(t))(\sin(\theta_1(t)) - \sin(\theta_2(t)))\end{aligned}$$

and hence we finally get

$$\dot{x}(t) = \begin{bmatrix} x_3(t) \\ x_4(t) \\ -\sin(x_1(t)) + \cos(x_1(t))(\sin(x_2(t)) - \sin(x_1(t))) + \frac{u(t)}{m\ell^2} \\ -\sin(x_2(t)) + \cos(x_2(t))(\sin(x_1(t)) - \sin(x_2(t))) \end{bmatrix}$$

2. Equilibria are found by setting $f(x(t)) = 0$. This immediately gives $x_3(t) = x_4(t) = 0$. Furthermore, we need the other two equations to also be satisfied. For all the variants above, this is also satisfied if $x_1(t) = x_2(t) = 0$.

3. **Grade according to student's solution to part 1**
Solution assuming spring force is given by $k(\theta_1 - \theta_2)$ **or** $k(\sin\theta_1 - \sin\theta_2)$**:**
Using the assumptions, we can write a linear version of the dynamics as

$$\dot{x}(t) = \begin{bmatrix} x_3(t) \\ x_4(t) \\ -x_1(t) + (x_2(t) - x_1(t)) + \frac{u(t)}{m\ell^2} \\ -x_2(t) + (x_1(t) - x_2(t)) \end{bmatrix}$$

which translates into the state-space form

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m\ell^2} \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t) + 0\end{aligned}$$

The matrices A, B, C, D can now be read off of the above equations.
Solution assuming spring force is given by $k\ell(\theta_1 - \theta_2)$ **or** $k\ell(\sin\theta_1 - \sin\theta_2)$**:**

Using the assumptions, we can write a linear version of the dynamics as

$$\dot{x}(t) = \begin{bmatrix} x_3(t) \\ x_4(t) \\ -x_1(t) + \ell(x_2(t) - x_1(t)) + \frac{u(t)}{m\ell^2} \\ -x_2(t) + \ell(x_1(t) - x_2(t)) \end{bmatrix}$$

which translates into the state-space form

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1-\ell & \ell & 0 & 0 \\ \ell & -1-\ell & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m\ell^2} \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t) + 0$$

The matrices A, B, C, D can now be read off of the above equations.

4. **Solution assuming spring force is given by** $k(\theta_1 - \theta_2)$ **or** $k(\sin\theta_1 - \sin\theta_2)$**:** We compute the observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$$

to obtain

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}.$$

If we swap rows 2 and 3 of this matrix (this does not change its rank), we get a lower triangular matrix with non-zero entries on the diagonal, which means it is full rank and the system is observable.

Solution assuming spring force is given by $k\ell(\theta_1 - \theta_2)$ **or** $k\ell(\sin\theta_1 - \sin\theta_2)$**:** We compute the observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$$

to obtain

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1-\ell & \ell & 0 & 0 \\ 0 & 0 & -1-\ell & \ell \end{bmatrix}.$$

which is full rank provided $\ell > 0$ for the same reasoning as in the above case.

5. As shown in Part 4, the system is observable. This means that with the given measurement of $\theta_1(t)$, we can build an observer that observes the entire state accurately.

Exercise 2

1	2	3	4	5	Exercise
3	5	6	6	5	25 Points

1. The A matrix is upper triangular, therefore, the eigenvalues are the diagonal elements. The set of eigenvalues of A is $\lambda = \{3,\, \alpha,\, \alpha\}$. For asymptotic stability, the real part of all eigenvalues of the system should be strictly negative. As $3 \notin \mathbb{R}_-$, there does not exist any α for which the system is asymptotically stable.

2. When only input $u_1(t)$ is applied to the system, we have the following dynamics

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 3 & 0 & -1 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix}}_{B_1} u_1(t). \tag{1}$$

The controllability matrix $\mathcal{C}_1 = \begin{bmatrix} B_1 & A\,B_1 & A^2\,B_1 \end{bmatrix}$ can be computed as

$$\mathcal{C}_1 = \begin{bmatrix} \beta & 3\beta & 9\beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

As the matrix $\mathcal{C}_1$ does not have a full rank of 3, there exists no value of β for which the system is controllable.

3. For the choice of parameters $\alpha = -1$, $\beta = -1$, the system dynamics are

$$\dot{x}(t) = \left(\begin{bmatrix} 3 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \right) x(t), \tag{3}$$

$$= \underbrace{\begin{bmatrix} 3-k_1 & -k_2 & -1-k_3 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}}_{A_{\text{cl}}} x(t). \tag{4}$$

As the closed loop matrix A_{cl} is upper triangular, the eigenvalues are given by $\lambda_{\text{cl}} = \{3-k_1,\, -1,\, -1\}$. For the closed-loop system to be asymptotically stable, the real part of the eigenvalues should be strictly negative. Therefore, for $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$, such that $k_1 > 3$, $k_2 \in \mathbb{R}$, $k_3 \in \mathbb{R}$, the closed loop system is asymptotically stable. However, it should be noted that only one of the closed-loop poles can be placed arbitrarily and the poles at -1 cannot be shifted.

4. When only input $u_2(t)$ is applied to the system, we have the following dynamics

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 3 & 0 & -1 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{B_2} u_2(t). \tag{5}$$

The controllability matrix $\mathcal{C}_2 = \begin{bmatrix} B_2 & A\,B_2 & A^2\,B_2 \end{bmatrix}$ can be computed as

$$\mathcal{C}_2 = \begin{bmatrix} 0 & -1 & -3-\alpha \\ 1 & \alpha+1 & \alpha^2+2\alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix}. \tag{6}$$

For controllability, the rank of $\mathcal{C}_2$ should be 3, or $\mathrm{Det}(\mathcal{C}_2) \neq 0$.

$$\mathrm{Det}(\mathcal{C}_2) = -1 \cdot (-\alpha^2 + 3\alpha + \alpha^2) + 1 \cdot (-\alpha^2 - 2\alpha + (3+\alpha)(1+\alpha)) = 3 - \alpha$$

Therefore, for $\alpha \neq 3$, system is controllable.

5. Consider the case when $\alpha = -1$, $\beta = -1$ and both the inputs $u_1(t)$, $u_2(t)$ are applied simultaneously. The system dynamics are given by

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 3 & 0 & -1 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} \beta & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{7}$$

The resulting controllability matrix $\mathcal{C}_3$ is composed of the elements of $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e.,

$$\mathcal{C}_3 = \begin{bmatrix} \beta & 0 & 3\beta & -1 & 9\beta & -3-\alpha \\ 0 & 1 & 0 & \alpha+1 & 0 & \alpha^2+2\alpha \\ 0 & 1 & 0 & \alpha & 0 & \alpha^2 \end{bmatrix}. \tag{8}$$

As the parameter value of $\alpha \neq 3$, the matrix $\mathcal{C}_3$ has full row rank already from the columns that are also present in $\mathcal{C}_2$ above and the system is controllable. The value of β is irrelevant for this. As the system is controllable, there exists a controller which can drive the states from any point in $\mathbb{R}^3$ to the origin in any given time. Therefore, the boss is correct.

Note that another way to solve this problem is to just say that since the system has already been shown to be controllable for the parameters with the input $u_2(t)$ alone, all that one has to do is set the input $u_1(t)$ to zero and reuse that result, no need for new computations. If a system is controllable, it can be steered from any state to any other state in any given time with the right input.

Exercise 3

1	2	3	4	Exercise
5	9	3	8	25 Points

1. The equilibria of the system are defined by

$$0 = x_2 \tag{9}$$
$$0 = -x_1 + \frac{1}{3}x_1^3 - x_2, \tag{10}$$

By solving for x_1 and x_2, we find that $\bar{x}_1 = (0,0), \bar{x}_{2,3} = (\pm\sqrt{3}, 0)$ are the equilibria of the system.

We linearize around $\bar{x}_{1,2,3}$ and obtain the following matrices, with

$$A_i = \frac{df}{dx}|_{x_i=\bar{x}_i}, i = \{1,2,3\}$$

defined by

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, A_{2,3} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix},$$

Either by explicitly calculating the eigenvalues of A_i or using the Hurwitz criterion, one can show that $\bar{x}_1$ is locally asymptotically stable and that both $\bar{x}_{2,3}$ are unstable.

2. There are two main alternatives for solving this question.

   (a) *Lasalle Argument* (Theorem 7.4) Note that

   - The set S is closed and bounded as seen graphically and hence compact.

   - The function $V(x)$ is well-defined and differentiable on the set S. The derivative along the system trajectories can be derived as

   $$\dot{V}(x) = \frac{1}{6}x_1^4 - \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2$$

   The derivative is negative definite on the set S. This can be seen as follows: If we look for the maximum of the derivative for any x_1 and x_2, we can see that the x_2 term is at most 0 independently of x_1. The x_1 term is then

   $$\frac{1}{6}x_1^4 - \frac{1}{2}x_1^2 = x_1^2\left(\frac{1}{6}x_1^2 - \frac{1}{2}\right)$$

   For the derivative to be always negative, the above has to be negative for all the admissible x_1. The x_1^2 in front is always positive, but the term in the brackets is negative as long as $|x_1| < \sqrt{3}$ as was given in the task.
   This proves the invariance of the set S.
   According to Lasalle Theorem, all the trajectories starting in S will converge to the largest invariant set M contained in the set $\{x \in S | \dot{V}(x) = 0\}$.
   We notice that $\forall x \in S$, $\bar{x}_1 = (0,0)$ is the only point that lies in the set M.
   Hence all the trajectories starting in S, will converge to the origin as $t \to \infty$.

(b) *Lyapunov Direct Method* (Theorem 7.3), where the following conditions are fulfilled for the set S described by the given Figure.

- One can guess that graphically, the function $V(x)$ takes positive values on the specified set S which contains the point $(0,0)$, so that $V(0,0) = 0$ and $V(x) \geq 0$, $\forall x \in S$.
- We take Lie derivative of $V(x)$ along the system trajectories, obtaining

$$\dot{V}(x) = \frac{1}{6}x_1^4 - \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2$$

We verify that the obtained derivative is negative definite on the set S as shown in the other solution approach. Note that graphically, one cannot guess if $V(x)$ is decreasing or not inside the set S.

Then, the origin is locally asymptotically stable $\forall x \in S$ according to the Direct Method of Lyapunov.

3. The origin cannot be globally asymptotically stable, since there are other equilibria than zero. If the system is initialized at one of these, it will not converge to zero.

4. The discretized system can be written as

$$x_{k+1} = x_k + \delta \begin{bmatrix} x_{2,k} \\ -x_{1,k} + \frac{1}{3}x_{1,k}^3 - x_{2,k} \end{bmatrix}$$

and if we then assume $\|x_k\|$ is small, we can approximate $x_{1,k}^3 \approx 0$ and deduce that

$$x_{k+1} = x_k + \delta \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x_k$$

which can be rewritten as

$$x_{k+1} = \begin{bmatrix} 1 & \delta \\ -\delta & 1-\delta \end{bmatrix} x_k \quad .$$

We can now compute the eigenvalues of the discretized system matrix:

$$\begin{aligned} \det\left(\lambda I - \begin{bmatrix} 1 & \delta \\ -\delta & 1-\delta \end{bmatrix}\right) &= (\lambda - 1)(\lambda - 1 + \delta) + \delta^2 \\ &= \lambda^2 + (\delta - 2)\lambda + (1 - \delta + \delta^2) \end{aligned}$$

which means the eigenvalues become

$$\lambda_{1,2} = -\frac{\delta}{2} \pm \sqrt{3}\frac{\delta}{2}i + 1$$

This in turn means that if $\delta = 1/2$, the eigenvalues become $\lambda_{1,2} = \frac{3}{4} \pm \frac{\sqrt{3}}{4}i$, which both have absolute value less than 1, meaning that the origin is asymptotically stable.

Alternatively, one can search for the range of $\delta > 0$, for which the system matrix has eigenvalues less than one. It must hold that $|1 + \delta\lambda_i| < 1$, where λ_i are the eigenvalues of the matrix A. It can be found that $|\frac{\delta^2}{4} + 1 - \delta + \frac{3}{4}| < 1$. From which follows that $0 < \delta < 1$. Hence for $\delta = \frac{1}{2}$, the origin of the discretized system is asymptotically stable.

We cannot distinguish between local and global asymptotic stability because if the linearized system is asymptotically stable, it is always globally asymptotically stable, whereas the original system may not be. This is because information is lost when the linearization is performed. In other words, the linearization is only accurate when its assumptions are satisfied, which they are not if one is away from the equilibrium.

Exercise 4

1	2	3	4	5	6	Exercise
4	7	5	5	2	2	25 Points

1. The state-space form of this system is:

$$\begin{aligned}\dot{x}_1(t) &= \dot{y}(t) = x_2(t) \\ \dot{x}_2(t) &= \ddot{y}(t) = -\frac{R}{m}\dot{y}(t) - \frac{k}{m}y(t) + \frac{1}{m}F(t)\end{aligned}$$

   and we hence get

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{R}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

2. We can use the standard formula to derive the transfer function $G(s)$:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{R}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \\ &= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix}}{s^2 + \frac{R}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{R}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \\ &= \frac{\frac{1}{m}}{s^2 + \frac{R}{m}s + \frac{k}{m}} \end{aligned}$$

   There are no pole-zero cancellations thus the system is both controllable and observable for all parameter values.

3. With the given parameter values, we get

$$G(s) = \frac{\frac{1}{10}}{s^2 + 0.2s + 1}$$

   which when compared to

$$G(s) = \frac{K\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

   yields $\omega_n = 1, \zeta = 0.1, K = 0.1$. Hence we have resonance since $\zeta = 0.1 < \frac{1}{\sqrt{2}}$ at $\omega = \sqrt{1 - 2 \cdot 0.01} \approx 1\,\mathrm{rad}/s$, with a magnitude of

$$|G(j\omega)| = \frac{\frac{1}{10}}{2 \cdot 0.1\sqrt{1 - 0.01}} \approx \frac{1}{2}$$

4. The Nyquist plot is shown in Figure 1. The unit circle is not drawn in the plot, but we see that everything is included in the unit circle.

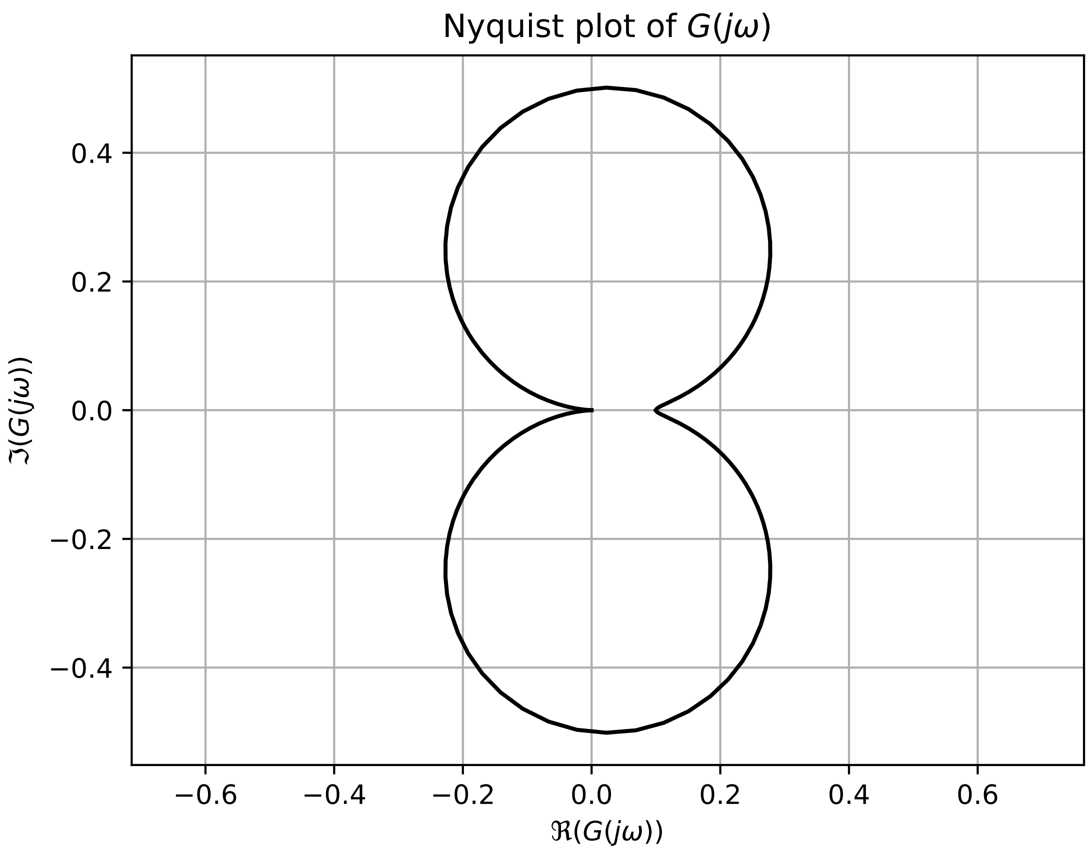

Figure 1: Nyquist plot of $G(s)$

5. From the plot, we can see that both gain as well as phase margin are infinite: The plot can either be scaled as much as we like and the point -1 will never become encircled (gain margin infinite), or the plot can be rotated as much as we like and -1 will not become encircled either.

6. No, combinations of gain and phase changes can easily make the point -1 encircled, rendering the system feedback unstable.