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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	Exercise
4	6	4	6	5	25 Points

1. Choose $x(t) = [V_c(t) \quad i_L(t) \quad \theta(t) \quad \dot{\theta}(t)]^T$. Energy in the system is stored in the current flowing through the inductor, voltage across the capacitor, and potential and kinetic energy of the pendulum mass.

2.
   - From the equation of the capacitor, $C\frac{dV_c}{dt} = -i_L$. Rearranging, $\mathbf{\frac{dV_c}{dt} = -\frac{1}{C}i_L}$.
   - Adding the voltages at the node at the top of the circuit, $V_c = i_L R + L\frac{di_L}{dt} + V_b$. Substituting for V_b and rearranging, we have that $\mathbf{\frac{di_L}{dt} = \frac{1}{L}(V_c - i_L R - k_2\dot{\theta})}$.
   - Obviously, $\mathbf{\frac{d\theta}{dt} = \dot{\theta}}$.
   - For the rotating mass, $I\ddot{\theta} = -\ell mg\sin(\theta) + k_1 i_L$. Since $I = m\ell^2$, we have that $\mathbf{\frac{d\dot{\theta}}{dt} = -\frac{g}{\ell}\sin(\theta) + \frac{k_1}{m\ell^2}i_L}$.

3. The above equations are linear except for the $sin(\theta)$ term. Substituting θ for $sin(\theta)$, we arrive at the linearized system

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{i}_L(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} & 0 & 0 \\ \frac{1}{L} & -\frac{R}{L} & 0 & -\frac{k_2}{L} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{k_1}{m\ell^2} & -\frac{g}{\ell} & 0 \end{bmatrix} \begin{bmatrix} V_c(t) \\ i_L(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$

4. We set $\det(\lambda I - A) = 0$. Now,

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda & 1 & 0 & 0 \\ -1 & \lambda & 0 & -k_2 \\ 0 & 0 & \lambda & -1 \\ 0 & -k_1 & 1 & \lambda \end{bmatrix} \\ &= \lambda\left(\lambda(\lambda^2+1) - k_2(k_1\lambda)\right) + (\lambda^2+1) \\ &= \lambda^4 + (1 - k_1k_2 + 1)\lambda^2 + 1 \\ &= \lambda^4 + (2 - k_1k_2)\lambda^2 + 1 \end{aligned}$$

   For an stable system, we want all roots of the determinant to have real part ≤ 0. Using the hint, we let $a_2 = 2 - k_1k_2$. Then, the condition that the system is unstable $\iff a_2 < 2$ implies instability for $2 - k_1k_2 < 2 \implies k_1k_2 > 0$. Thus, the system is stable if $k_1k_2 \leq 0$, which means that k_1 and k_2 have opposing signs (or at least one is zero).

5. If $R = 0$, the eigenvalues of the system are all purely imaginary. The system oscillates at several frequencies, with no energy loss. If R is nonzero, then there is a dissipative component in the system, and the oscillations would be damped.

Exercise 2

1	2	3	4	5	Exercise
4	6	5	6	4	25 Points

1. Using $x(t) = S^{-1}\hat{x}(t)$, we can write:

$$\dot{\hat{x}}(t) = S\dot{x}(t) = SAS^{-1}\hat{x}(t) + SBu(t)$$
$$y(t) = CS^{-1}\hat{x}(t)$$

which means we get:

$$\hat{A} = SAS^{-1}, \quad \hat{B} = SB, \quad \hat{C} = CS^{-1}.$$

2. We can simply compute the eigenvalue decomposition of A:

$$A = V\Lambda V^{-1}$$

where Λ is the diagonal matrix of eigenvalues of A. First we get the eigenvalues:

$$\det(\lambda I - A) = \lambda^2 + 7\lambda + 12 \stackrel{!}{=} 0$$

which means we get $\lambda_1 = -3, \lambda_2 = -4$ as eigenvalues. For V, we compute the eigenvectors:

$$Av \stackrel{!}{=} \lambda_i v$$

For λ_1, we get that $v_2 = -3v_1$ (scaling is irrelevant). For λ_2, we get $v_2 = -4v_1$. Hence finally we can write

$$V = \begin{bmatrix} 1 & 1 \\ -3 & -4 \end{bmatrix}$$

and

$$V^{-1} = \begin{bmatrix} 4 & 1 \\ -3 & -1 \end{bmatrix}$$

The desired state transformation is then $S = V^{-1}$. For the state transition matrix, we get:

$$e^{At} = Ve^{\Lambda t}V^{-1} = \begin{bmatrix} -3e^{-4t} + 4e^{-3t} & -e^{-4t} + e^{-3t} \\ 12e^{-4t} - 12e^{-3t} & 4e^{-4t} - 3e^{-3t} \end{bmatrix}.$$

3. The output impulse response is given by:

$$K(t) = Ce^{At}B + D\delta(t)$$

and we have $D = 0$. Using the result above for e^{At}, we can write:

$$K(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -3e^{-4t} + 4e^{-3t} & -e^{-4t} + e^{-3t} \\ 12e^{-4t} - 12e^{-3t} & 4e^{-4t} - 3e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{-3t} - e^{-4t}$$

The answer will look the same no matter what coordinates are chosen, since it is still the same system and the input and output are still in the same coordinates, only the state changed coordinates.

4. We first introduce variables for $\hat{Q}$ and rewrite the equation system:

$$\begin{bmatrix} 0 & -12 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} = \begin{bmatrix} -24 & 0 \\ 0 & -12 \end{bmatrix}$$

This system gives us 3 equations:

$$\begin{aligned} -24 &= -12q_2 + -12q_2 \\ 0 &= -12q_3 + q_1 - 7q_2 \\ -12 &= q_2 - 7q_3 + q_2 - 7q_3 \end{aligned}$$

From which we get (in this order) $q_2 = 1, q_3 = 1, q_1 = 19$.

5. If we used $\hat{A}$, we would get a diagonal $\hat{Q}$ due to $\hat{A}$ only scaling $\hat{Q}$ and R dictating the offdiagonal entries having to be 0. We expect the Q to be positive semidefinite and unique nonetheless, because the system is asymptotically stable (eigenvalues are -3 and -4) and both A and $\hat{A}$ are invertible.

Exercise 3

1	2	3	4	5	Exercise
5	6	4	6	4	25 Points

1. We verify the two required properties:

$$e^{\bar{A}0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

For the second property:

$$\frac{d}{d} t e^{\bar{A}t} = \begin{bmatrix} -e^{-t} & e^{-t} - te^{-t} \\ 0 & -e^{-t} \end{bmatrix}$$

but also

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} e^{\bar{A}t} = \begin{bmatrix} -e^{-t} & e^{-t} - te^{-t} \\ 0 & -e^{-t} \end{bmatrix}$$

which verifies that the state transition matrix as given in the task is correct. Since we can immediately see that the eigenvalues are $\lambda_1 = \lambda_2 = -1$, we know that the system is asymptotically stable (and therefore also stable). The controllability matrix is given by

$$P = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

which has full rank (determinant is -1), hence the system is controllable.

2. The matrix A is given by

$$A = e^{\bar{A}T} = \begin{bmatrix} e^{-T} & Te^{-T} \\ 0 & e^{-T} \end{bmatrix}$$

whereas we can compute B using:

$$\begin{aligned}
B &= \int_0^T e^{\bar{A}(T-\tau)} \bar{B} d\tau \\
&= \int_0^T \begin{bmatrix} e^{-(T-\tau)} & (T-\tau)e^{-(T-\tau)} \\ 0 & e^{-(T-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \\
&= \begin{bmatrix} \int_0^T (T-\tau)e^{-(T-\tau)} d\tau \\ \int_0^T e^{-(T-\tau)} d\tau \end{bmatrix} = \begin{bmatrix} \int_0^T (T-\tau)e^{\tau-T} d\tau \\ \int_0^T e^{\tau-T} d\tau \end{bmatrix} \\
&= \begin{bmatrix} Te^{-T} \int_0^T e^{\tau} d\tau - e^{-T} \int_0^T \tau e^{\tau} d\tau \\ e^{-T} \int_0^T e^{\tau} d\tau \end{bmatrix} \\
&= \begin{bmatrix} T(1 - e^{-T}) - e^{-T} \left[e^{\tau}(\tau - 1) \right]_0^T \\ e^{-T}(e^T - 1) \end{bmatrix} \\
&= \begin{bmatrix} T(1 - e^{-T}) - (T-1) - e^{-T} \\ e^{-T}(e^T - 1) \end{bmatrix} = \begin{bmatrix} 1 - e^{-T} - Te^{-T} \\ 1 - e^{-T} \end{bmatrix}
\end{aligned}$$

3. It is asymptotically stable for all of them, since $e^{-T} < 1$ for all positive T. As for controllability, we first define $a = e^{-T}$ and we then get that

$$\begin{aligned} P = \begin{bmatrix} B & AB \end{bmatrix} &= \begin{bmatrix} (1-a-Ta) & a(1-a-Ta)+Ta(1-a) \\ 1-a & a(1-a) \end{bmatrix} \\ &= \begin{bmatrix} (1-a-Ta) & a-a^2+Ta-2Ta^2 \\ 1-a & a-a^2 \end{bmatrix} \end{aligned}$$

The determinant of this is then:

$$\begin{aligned} \det(P) &= (1-a-Ta)(a-a^2) - (1-a)(a-a^2+Ta-2Ta^2) \\ &= a - a^2 - Ta^2 - (a^2 - a^3 - Ta^3) - (a - a^2 + Ta - 2Ta^2) \\ &\quad + (a^2 - a^3 + Ta^2 - 2Ta^3) \\ &= -Ta(a^2 - 2a + 1) = -Ta(a-1)^2 \end{aligned}$$

Since we know that $T > 0$ and $0 < a < 1$, the determinant can never be 0 and the system is controllable for all $T > 0$.

4. We have that

$$\begin{aligned} x_{k+1} &= x_k + T(\bar{A}x_k + \bar{B}u_k) \\ &= (I + T\bar{A})x_n + T\bar{B}u_k \end{aligned}$$

hence we have that

$$\tilde{A} = \begin{bmatrix} 1-T & T \\ 0 & 1-T \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} 0 \\ T \end{bmatrix}$$

5. The eigenvalues are $1-T$, hence the system will have eigenvalues with absolute value less than 1 (making it asymptotically stable) for $0 < T < 2$. For controllability, we have that

$$P = \begin{bmatrix} 0 & T^2 \\ T & T - T^2 \end{bmatrix}$$

which has determinant $-T^3$. That means the system is controllable for all $T > 0$.

6. $T = 1$ makes $\tilde{A}$ nilpotent. This would suggest that the discrete time system would converge to zero in just 2 steps if one input is applied. However, from our answer in Part 2 we know that is not the case for the sampled data system, that converges to zero asymptotically at the rate e^{-1}.

Exercise 4

1	2	3	4	5	Exercise
3	4	5	7	6	25 Points

1. The transfer function is given by $G(s) = C(sI - A)^{-1}B$, for the system Σ_1 this results in

$$\begin{aligned} G_1(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ -2 & s-a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{(s+1)(s-a)} \begin{bmatrix} s-a & 0 \\ 2 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ &= \frac{2}{(s+1)(s-a)}. \end{aligned}$$

2. Because there are no pole-zero cancellations for any value of a the system is observable and controllable for all values of a. This allows us to draw conclusions about stability of the system Σ_1 based on its transfer function $G_1(s)$. The poles of the transfer function are $p_1 = -1$ and $p_2 = a$. We conclude that for $a < 0$ it holds that $\Re\{p_i\} < 0$ for $i \in \{1, 2\}$ and the system is asymptotically stable. This is true regardless of whether or not the matrix A is diagonalizable. For $a = 0$ the poles are distinct, i.e., $p_1 = -1$ and $p_2 = 0$, it follows that the matrix A is diagonalizable and the system is stable because of $\Re\{p_2\} = 0$. Because asymptotic stability implies stability, the system is not only stable for $a = 0$ but for $a \leq 0$. Moreover, it is unstable for $a > 0$ because $\Re\{p_2\} > 0$.

3. The magnitude of $G_1(s)$ with $a = -1$ is given by

$$\begin{aligned} \|G_1(j\omega)\| &= \|\frac{2}{(j\omega+1)^2}\|, \\ &= \frac{2}{\|j\omega+1\|^2}, \\ &= \frac{2}{\sqrt{\omega^2+1}^2}, \\ &= \frac{2}{\omega^2+1}. \end{aligned}$$

It directly follows that $\|G_1(j\omega^\star)\| = 1$ holds for $\omega^\star = -1$ and $\omega^\star = 1$, we are only interested in the positive solution. To compute the phase at $\omega = 1$, we first compute $G_1(j\omega)$ for $a = -1$ and $\omega = \omega^\star = 1$. This results in

$$G_1(j\omega^\star) = \frac{2}{(j+1)^2} = -j.$$

In other words, for $a = -1$ it holds that $\Re\{G_1(j\omega^\star)\} = 0$ and $\Im\{G_1(j\omega^\star)\} = -1$, i.e. the phase angle at $\omega = \omega^\star$ is $-90°$. By inspecting the phase margin in the two Bode plots we conclude that Bode plot (a) shows the system Σ_1 with $a = -1$.

4. The transfer function $G_1(s)$ has an unstable pole at $s = \frac{1}{4}$. Therefore, the variable P in the lecture notes of the Nyquist stability criterion is one. This implies that for stability of the closed loop system we need to satisfy the equation $N = -P = -1$, where N denotes the number of clockwise encirclements of the Nyquist curve around the critical point $-\frac{1}{K}$. Therefore we require one counter-clockwise encirclement of the point $-\frac{1}{K}$ to ensure stability of the closed loop. By considering the Nyquist diagram shown in Figure 1 this is ensured when $-\frac{1}{K} < 0$, and $-\frac{1}{K} > -8$ hold. In other words, the closed-loop is stable for $K > \frac{1}{8}$.

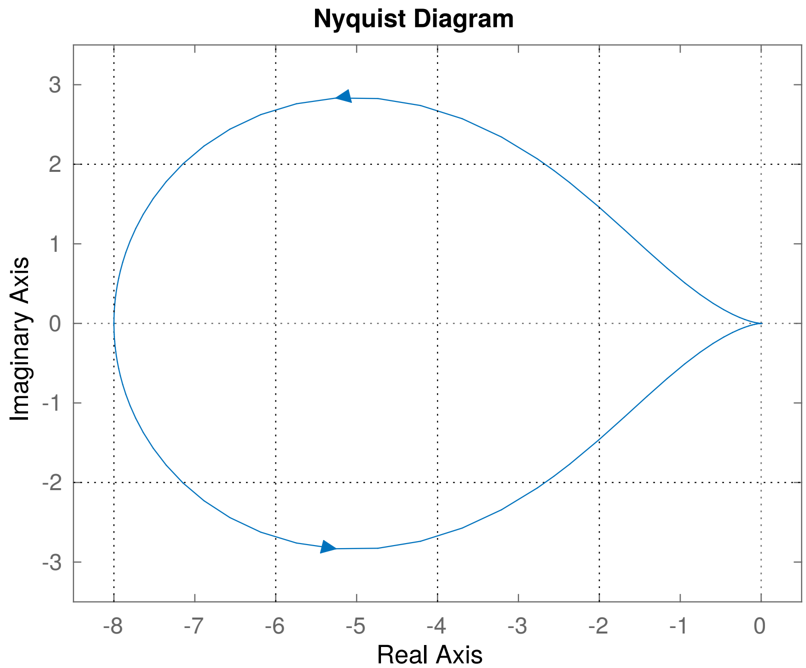

Figure 1: Nyquist diagram of the open-loop transfer function of Σ_1.

5. The transfer function of the closed loop can be computed as follows:

$$\begin{aligned} G(s) &= \frac{G_2(s)G_1(s)}{1 + G_2(s)G_1(s)}, \\ &= \frac{\frac{1}{2}\frac{s-\frac{1}{4}}{s+3}\frac{2}{(s+1)(s-\frac{1}{4})}}{1 + \frac{1}{2}\frac{s-\frac{1}{4}}{s+3}\frac{2}{(s+1)(s-\frac{1}{4})}}, \\ &= \frac{\frac{1}{(s+3)(s+1)}}{1 + \frac{1}{(s+3)(s+1)}}, \\ &= \frac{1}{s^2 + 4s + 4} = \frac{1}{(s+2)^2}. \end{aligned}$$

The transfer function $G(s)$ is stable. However, there is a pole-zero cancellation of the unstable pole at $\frac{1}{4}$. Any mismatch in the location of this pole and the corresponding zero in the transfer function of the controller will result in an unstable closed-loop system. Therefore, because the parameters of the system are never known exactly, this controller will, in practice, not stabilize the system Σ_1.