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Signal and System Theory II
4. Semester, BSc

Solutions

Solution for exercise 1

1	2	3	4	5	Exercise
6	3	6	4	6	25 Points

1. The system can be written as

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{v_{\text{in}}}{A_1} - \frac{C_v u_1(t)}{A_1}\sqrt{2gx_1(t)} \\ \frac{C_v u_1(t)}{A_2}\sqrt{2gx_1(t)} - \frac{C_v u_2(t)}{A_2}\sqrt{2gx_2(t)} \end{bmatrix} \tag{1}$$

2. For the steady state, the derivatives are 0 and we exchange $x_1(t)$ with α_1 as well as $x_2(t)$ with α_2:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{v_{\text{in}}}{A_1} - \frac{C_v \overline{u_1}}{A_1}\sqrt{2g\alpha_1} \\ \frac{C_v \overline{u_1}}{A_2}\sqrt{2g\alpha_1} - \frac{C_v \overline{u_2}}{A_2}\sqrt{2g\alpha_2} \end{bmatrix} \tag{2}$$

which implies

$$\overline{u_1} = \frac{v_{\text{in}}}{C_v\sqrt{2g\alpha_1}}.$$

With this, the second equation becomes

$$0 = \frac{v_{\text{in}}}{A_2} - \frac{C_v \overline{u_2}}{A_2}\sqrt{2g\alpha_2}$$

which means

$$\overline{u_2} = \frac{v_{\text{in}}}{C_v\sqrt{2g\alpha_2}}.$$

3. The matrices are as follows:

$$A = \begin{bmatrix} -\frac{C_v\sqrt{2g}u_1(t)}{A_1}\frac{1}{2\sqrt{x_1(t)}} & 0 \\ \frac{C_v\sqrt{2g}u_1(t)}{A_2}\frac{1}{2\sqrt{x_1(t)}} & -\frac{C_v\sqrt{2g}u_2(t)}{A_2}\frac{1}{2\sqrt{x_2(t)}} \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{C_v}{A_1}\sqrt{2gx_1(t)} & 0 \\ \frac{C_v}{A_2}\sqrt{2gx_1(t)} & -\frac{C_v}{A_2}\sqrt{2gx_2(t)} \end{bmatrix}$$

Putting in the constants from the previous task leads to:

$$A = \begin{bmatrix} -\frac{v_{\text{in}}}{2A_1\alpha_1} & 0 \\ \frac{v_{\text{in}}}{2A_2\alpha_1} & -\frac{v_{\text{in}}}{2A_2\alpha_2} \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{C_v}{A_1}\sqrt{2g\alpha_1} & 0 \\ \frac{C_v}{A_2}\sqrt{2g\alpha_1} & -\frac{C_v}{A_2}\sqrt{2g\alpha_2} \end{bmatrix}$$

4. The system is stable: Because all the constants are > 0, the eigenvalues have real parts smaller than 0 (we can read them from the diagonal of A). This means the non-linear system is locally asymptotically stable around this equilibrium.

5. Since the system does not have any unstable poles (because it is stable), we need the point $-1/K$ to be encircled 0 times for stability. The inadmissible K are therefore in the range $-1/K \in [-\frac{1}{3}, 0)$ (approximately), meaning $K \geq 3$ renders the system unstable. Since the question asks about when the stability properties change (i.e. when it becomes unstable), the range of K asked for is $K \geq 3$.

Solution for exercise 1

1	2	3	4	5	Exercise
X	X	X	X	X	25 Points

1. Setting $det(\lambda I - A) = 0$,

$$det\left(\begin{bmatrix}\lambda & -a \\ a & \lambda\end{bmatrix}\right) = 0$$

Thus, the characteristic equation is $\lambda^2+a^2 = 0$, and the eigenvalues are $\lambda = \pm aj$. $\lambda = +aj$ has an eigenvector of $[1\ j]^T$, while $\lambda = -aj$ has an eigenvector of $[j\ 1]^T$. Since the eigenvalues have linearly independent eigenvectors, the matrix is diagonalizable.

2. We write $A = W\Lambda W^{-1}$, with W the matrix of eigenvectors and Λ the matrix with the eigenvalues on the diagonal.

$$A = \begin{bmatrix}1 & j \\ j & 1\end{bmatrix}\begin{bmatrix}aj & 0 \\ 0 & -aj\end{bmatrix}\begin{bmatrix}1 & j \\ j & 1\end{bmatrix}^{-1}$$

As we saw in lecture, $e^{At} = W\Lambda^t W^{-1}$. Thus,

$$e^{At} = \begin{bmatrix}1 & j \\ j & 1\end{bmatrix}\begin{bmatrix}e^{ajt} & 0 \\ 0 & e^{-ajt}\end{bmatrix}\begin{bmatrix}1 & j \\ j & 1\end{bmatrix}^{-1} = 1/2\begin{bmatrix}e^{ajt}+e^{-ajt} & -je^{ajt}+je^{-ajt} \\ je^{ajt}-je^{-ajt} & e^{ajt}+e^{-ajt}\end{bmatrix} = \begin{bmatrix}\cos(at) & \sin(at) \\ -\sin(at) & \cos(at)\end{bmatrix}$$

3. Since A is diagonalizable, with eigenvalues that have zero real parts, the system is stable, but not asymptotically stable, for all a.

4. To determine controllability, we form the controllability matrix $P = \begin{bmatrix}B & AB\end{bmatrix} = \begin{bmatrix}b_1 & ab_2 \\ b_2 & -ab_1\end{bmatrix}$. Looking at the determinant of this matrix, P is invertible iff $-ab_1^2 - ab_2^2 \neq 0$. If $a = 0$, then the system is not controllable for any b_1, b_2. If $a \neq 0$, the system is controllable iff $b_1 \neq 0$, $b_2 \neq 0$.

5. Since $x_0 = [0\ 0]^T$, $\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D = \frac{s}{s^2+a^2} + 1$. Since we want the step response of this system, $U(s) = 1/s$, and thus $Y(s) = \frac{1}{s^2+a^2} + \frac{1}{s}$. Taking the inverse Laplace transform, $y(t) = 1 + \frac{1}{a}\sin(at)$.

Exercise 3

1	2	3	4	5	Exercise
5	3	5	5	7	25 Points

1.

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$D = 0$$

2.

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{8} & \frac{3}{4} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$D = 0$$

3. a) Stability: the eigenvalues of the system matrix A are $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = \frac{1}{2}$. Since the magnitudes of both eigenvalues are strictly smaller than 1, the autonomous system is asymptotically stable.

b) Controllability: the controllability matrix P has full rank and thus the system is controllable.

$$P = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & \frac{3}{2} \end{pmatrix}$$

c) Observability: the observability matrix O has full rank and thus the system is observable

$$O = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{8} & \frac{3}{4} \end{pmatrix}$$

4. No, since the system matrix A does not have both eigenvalues at zero, it is not nilpotent and the ZIT $x_k = A^{k-N} x_N \quad k = N, N+1, \dots$ does not reach $[0\ 0]^\top$ in a finite number of steps and remain there.

5. Since the pair (A, B) is controllable, there exists a state feedback law $u_k = F_k x_k$ such that the closed-loop system matrix $A + BF$ has deadbeat behavior. Therefore, we have to find $F = [f_1\ f_2] \in \mathbb{R}^{1x2}$ such that the matrix $A + BF$ has both eigenvalues at zero. We can achieve this by computing the characteristic polynomial of $A + BF$ as a function of f_1 and f_2 and comparing coefficients:

$$A + BF = \begin{pmatrix} 0 & 1 \\ -\frac{1}{8} + 2f_1 & \frac{3}{4} + 2f_2 \end{pmatrix}$$

$$det\big((A + BF) - \lambda I\big) = \lambda^2 - \lambda \cdot (\frac{3}{4} + 2f_2) - 2f_1 + \frac{1}{8}$$

By comparing coefficients with the polynomial $1\lambda^2 + 0\lambda + 0$ (i.e. both eigenvalues of $A + BF$ at zero), we obtain $f_1 = \frac{1}{16}$ and $f_2 = -\frac{3}{8}$ and

$$F = \begin{pmatrix} f_1 & f_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{16} & -\frac{3}{8} \end{pmatrix}$$

Sanity check:

$$A + BF = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is indeed nilpotent.

Exercise 4

1	2	3	4	5	Exercise
7	5	5	3	5	25 Points

1. a) State-space form is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} x \\ \frac{1}{m}(-c(x_1) - d(x_2)) \end{bmatrix} \tag{3}$$

   b) It depends on the functions $c(x)$ and $d(\dot{x})$:
   If they are linear (for example $c(x) = kx$ and $d(\dot{x}) = g\dot{x}$), the system is linear, as it can be written as $\dot{x} = Ax$ with constant coefficient A (for example $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{g}{m} \end{bmatrix}$).
   If one or both of $c(x)$ and $d(\dot{x})$ are nonlinear, the system is also nonlinear, as it cannot be written as $\dot{x} = Ax$ with constant coefficient A.

   c) No, the system is time-**in**variant, as there is no time-dependency: $\dot{x} = f(x) \neq f(x, t)$.

   d) Yes, the system is autonomous, as there is no input to the system, i.e. $\dot{x} = f(x) \neq f(x, u)$.

2. At an equilibrium point, it holds:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 = 0 \\ c(x_1) - \underbrace{d(x_2 = 0)}_{0} = 0 \end{bmatrix} \Rightarrow c(x_1) = 0, x_1 = 0. \tag{4}$$

   The only equilibrium point of the system is the origin $(0, 0)$.

3. We use Theorem 7.2 from the Lecture Notes, with the Lyapunov function $V = \frac{1}{2}mx_2^2 + \int_0^{x_1} c(s)ds$ (energy of the system) and $S = \mathbb{R}^2$:

   i) $V(0, 0) = 0$

   ii) $V(x) > 0,\ \forall x \in \mathbb{R}^2 \setminus (0, 0)$

   iii) $\dot{V}(x) = m\, x_2\, \dot{x}_2 + c(x_1)\, \dot{x}_1 = m\, x_2\, \frac{1}{m}\, (-c(x_1) - d(x_2)) + c(x_1)\, x_2 = -x_2\, d(x_2) \leq 0,\ \forall x \in \mathbb{R}^2$

4. Theorem 7.3 from the Lecture Notes, with the Lyapunov function candidate $V = E = \frac{1}{2}mx_2^2 + \int_0^{x_1} c(s)ds$ (as before) and $S = \mathbb{R}^2$:

   i) $V(0, 0) = 0$

   ii) $V(x) > 0,\ \forall x \in \mathbb{R}^2 \setminus (0, 0)$

   iii) $\dot{V}(x) = -x_2\, d(x_2) \overset{?}{<} 0,\ \forall x \in \mathbb{R}^2$? No, this is not true! As $\dot{V}(x) = 0$ for $x_2 = 0$

As iii) does not hold, asymptotic stability cannot be shown.

5. Can you use LaSalle (Theorem 7.4 in the Lecture Notes) to show global asymptotic stability of the origin?

    i) Take $S = \{x \in \mathbb{R}^2 | V(x) \leq \xi\}$, with any $\xi \in \mathbb{R}^+ \setminus \{0\}$
    Then, $V(0) = 0, V(x) > 0, \forall x \in S \setminus 0$ and S is closed and bounded and thus compact.

    ii) $\dot{V}(x) = -x_2\ d(x_2) \leq 0,\ \forall x \in S\ \Rightarrow S$ invariant

    iii) $\bar{S} = \{x \in S | \dot{V}(x(t)) = 0\}$, given as $\bar{S} = \{x \in \mathbb{R}^2 | x_2 = 0\}$

    iv) We show, that the largest invariant set M in $\bar{S}$ is the origin:
    $x_2 = 0 \Rightarrow \dot{x}_1 = 0 \Rightarrow c(x_1) + 0 = 0 \Leftrightarrow x_1 = 0$

    v) By Theorem 7.4: All trajectories starting in S approach $M = \{(0,0)\}$ at $t \to \infty$.

Due to stability of the system (Theorem 7.2) from part 4. from this exercise and LaSalle (Theorem 7.4), we know that for $S = \mathbb{R}^2$, the origin $x = (0,0)$ is globally asymptotically stable.