Automatic Control Laboratory
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D-ITET
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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	Exercise
12	13	25 Points

1. (a) The impulse response is given by (2p)

$$K(t) = C\Phi(t)B + D\delta(t) = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}e^{-2t} & 0 \\ 0 & e^{-3t}\end{pmatrix}\begin{pmatrix}1 \\ 1\end{pmatrix} = e^{-2t} + 2e^{-3t}.$$

Both poles of the system appear in the impulse response. (1p) The laplace transform of the impulse response is the transfer function. As both poles appear in the transfer function, we can conclude that there is no pole-zero-cancellation (1p) and therefore, the system is completely observable and controllable. (1p)

(b) The transfer function $G(s)$ is the Laplace-transform of the impulse response and given by (3p)

$$G(s) = \mathcal{L}(K(t)) = \frac{1}{s+2} + \frac{2}{s+3} = \frac{3s+7}{s^2+5s+6}$$

Alternative solution: The transfer function can directly be computed, as the system matrices are known.

$$\begin{aligned} G(s) &= C(sI-A)^{-1}B \\ &= \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+3}\end{pmatrix}\begin{pmatrix}1 \\ 1\end{pmatrix} \\ &= \frac{1}{s+2} + \frac{2}{s+3} \\ &= \frac{3s+7}{s^2+5s+6} \end{aligned}$$

(c)

$$\begin{aligned} y(t) &= C\Phi(t-t_0)x(t_0) + \int_{t_0}^{t} K(\tau)u(t-\tau)d\tau \\ &= \int_0^t 2(e^{-2\tau} + 2e^{-3\tau})d\tau \\ &= \left[2e^{-2\tau} + 4e^{-3\tau}\right]_0^t \\ &= -e^{-2t} + -\frac{4}{3}e^{-3t} + \frac{7}{3} \quad (4p) \end{aligned}$$

2. (a) The characteristical polynomial is given by

$$(s+1)(s-2)(s+3) = s^3 + 2s^2 - 5s - 6 = s^3 + a_1 s^2 + a_2 s + a_3 \quad (2p)$$

and therefore the observable canonical form reads the following:

$$A = \begin{pmatrix}0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1\end{pmatrix} = \begin{pmatrix}0 & 0 & 6 \\ 1 & 0 & 5 \\ 0 & 1 & -2\end{pmatrix}, \quad B = \begin{pmatrix}b_3 \\ b_2 \\ b_1\end{pmatrix} = \begin{pmatrix}-1 \\ 1 \\ 0\end{pmatrix}, \quad C = \begin{pmatrix}0 & 0 & 1\end{pmatrix}. \quad (2p)$$

(b) The differential equation for the error dynamics is given by

$$\begin{aligned}\dot{e}(t) = &\quad \dot{x}(t) - \dot{\tilde{x}}(t)\\ = &\quad Ax(t) + Bu(t) - [A\tilde{x}(t) + Bu(t) + Ly - LC\tilde{x}]\\ = &\quad A(x(t) - \tilde{x}) - LC(x(t) - \tilde{x}(t)\\ = &\quad (A - LC)e(t). \quad (3p)\end{aligned}$$

(c) The desired polynomial is given by

$$(s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6. \quad (1p)$$

The A-matrix of the system with observer is

$$A - \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = A - \begin{pmatrix} 0 & 0 & l_1 \\ 0 & 0 & l_2 \\ 0 & 0 & l_3 \end{pmatrix}. \quad (2p)$$

One can see, that it is still in canonical observable form, (1p) which makes the computation of the observer gains very easy

$$\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} p_1 - a_3 \\ p_2 - a_2 \\ p_3 - a_1 \end{pmatrix} = \begin{pmatrix} 6+6 \\ 11+5 \\ 6-2 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \\ 4 \end{pmatrix}, \quad (2p)$$

with p_i the coefficients of the desired characteristical polynomial.
For the coefficient matching, you can also compute:

$$s^3 + (l_3 + 2)s^2 + (l_2 - 5)s + (l_1 - 6)$$

Exercise 2

1	2	3	4	5	Exercise
4	6	5	7	3	25 Points

1. The eigenvalues of the matrix are $\lambda_{1,2} = 1$ and $\lambda_3 = \frac{1}{2}$. They are independent of the value of a. Because there exist eigenvalues on the unit circle, the system is not asymptotically stable.

2. The controllability matrix is given as

$$P = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 0 & -2 & -4 \\ 0 & 2a & 5a \end{pmatrix}.$$

For $a = 0$, the set of reachable states is given as

$$\text{range}(P) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

For $a \neq 0$ we have

$$\det(P) = 2(-10a + 8a) = -4a \neq 0,$$

hence, the system is controllable for $a \neq 0$ and all states are reachable.

3. The observability matrix is given as

$$Q = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}.$$

Obviously, $\det(Q) = 0$ for all values of a, hence, the system is never observable.

4. The system is unobservable for all values of a, hence, there will be at least one pole-zero cancellation in the transfer function, i.e. the transfer function will have less than three poles.

To determine whether the transfer function has one or two pole-zero-cancellations, we have to determine for which values of λ the matrices

$$\begin{pmatrix} C \\ \lambda I - A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & \lambda I - A \end{pmatrix}$$

lose rank.

It is easy to see that

$$\begin{pmatrix} C \\ \lambda I - A \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda - 1 & -1 & 0 \\ 1 & \lambda & 0 \\ -a & 0 & \lambda - \frac{1}{2} \end{pmatrix}$$

loses rank for $\lambda = \frac{1}{2}$ for all values of a (the last column is identically zero). Similarly, the matrix

$$\begin{pmatrix} B & \lambda I - A \end{pmatrix} = \begin{pmatrix} 2 & \lambda - 1 & -1 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & -a & 0 & \lambda - \frac{1}{2} \end{pmatrix}$$

only loses rank for $a = 0$ and $\lambda = \frac{1}{2}$ (the last row is identically zero).

Hence, the only mode being cancelled in the transfer function for all values of a is $\lambda_3 = \frac{1}{2}$ and the transfer function will always have exactly two poles $\lambda_{1,2} = 1$.

5. From the controllability matrix P in part 1 it is clear that only adding

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_3(k)$$

would make the system controllable for all values of a (P is full rank).

Exercise 3

1	2	3	4	5	6	Exercise
4	3	3	5	3	7	25 Points

1. The differential equation is

$$m\ddot{z}(t) = mg - C\frac{u(t)}{(z(t)+\bar{z})^2}$$

[2 points]. It is not linear **[1 point]**, but it is time invariant **[1 point]**.

2. In order to find the equilibrium, we simply set $z(t)$ and all its derivatives to zero, leading to **[2 points]**

$$0 = mg - C\frac{\bar{u}}{\bar{z}^2} \iff \bar{u} = \frac{mg\bar{z}^2}{C}$$

3. The state-space form of the system is given as **[3 points]**

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ g - \frac{C}{m}\frac{u(t)}{(x_1(t)+\bar{z})^2} \end{bmatrix}$$
$$y(t) = x_1(t)$$

4. The linearized system equations with $x(t) = \begin{bmatrix} z(t) & \dot{z}(t) \end{bmatrix}^T$ are

$$\frac{d}{dt}\delta x(t) = \left.\begin{bmatrix} 0 & 1 \\ \frac{2Cu(t)}{m(z(t)+\bar{z})^3} & 0 \end{bmatrix}\right|_{\substack{z(t)=0 \\ u(t)=\bar{u}}} \delta x(t) + \left.\begin{bmatrix} 0 \\ -\frac{C}{m(z(t)+\bar{z})^2} \end{bmatrix}\right|_{\substack{z(t)=0 \\ u(t)=\bar{u}}} \delta u(t)$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{2C\bar{u}}{m\bar{z}^3} & 0 \end{bmatrix} \delta x(t) + \begin{bmatrix} 0 \\ -\frac{C}{m\bar{z}^2} \end{bmatrix} \delta u(t)$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{2g}{\bar{z}} & 0 \end{bmatrix} \delta x(t) + \begin{bmatrix} 0 \\ -\frac{C}{m\bar{z}^2} \end{bmatrix} \delta u(t)$$

Grading: **[2 points]** for A matrix, **[1 point]** point for B matrix, **[2 points]** for properly substituting equilibrium point values.

5. Since $\bar{u}, C, \bar{z} > 0$ holds, the eigenvalues given by the characteristic equation

$$\lambda^2 - \frac{2g}{\bar{z}} = 0$$

are $\lambda_{1,2} = \pm\sqrt{\frac{2g}{\bar{z}}}$ **[1 point]**, and the considered equilibrium point is unstable **[1 point]**. That means a disturbance will drive it away from there, making it useless as a suspension system with an unknown force acting on the wheel **[1 point]**.

6. Applying the input $\delta u(t) = k_1 x_1(t) + k_2 x_2(t)$ to the linearized system yields

$$\begin{aligned}
\frac{d}{dt}\delta x(t) &= \begin{bmatrix} 0 & 1 \\ \frac{2g}{\bar{z}} & 0 \end{bmatrix} \delta x(t) + \begin{bmatrix} 0 \\ -\frac{C}{m\bar{z}^2} \end{bmatrix} \delta u(t) \\
&= \begin{bmatrix} 0 & 1 \\ \frac{2g}{\bar{z}} & 0 \end{bmatrix} \delta x(t) + \begin{bmatrix} 0 \\ -\frac{C}{m\bar{z}^2} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \delta x(t) \\
&= \begin{bmatrix} 0 & 1 \\ \frac{2g}{\bar{z}} - \frac{Ck_1}{m\bar{z}^2} & -\frac{Ck_2}{m\bar{z}^2} \end{bmatrix} \delta x(t)
\end{aligned}$$

[2 points] for equations, **[1 point]** for arriving at a form $\delta \dot{x}(t) = A\delta x(t)$. The characteristic polynomial of the state transition matrix is

$$\lambda^2 + \frac{Ck_2}{m\bar{z}^2}\lambda + \left(\frac{Ck_1}{m\bar{z}^2} - \frac{2g}{m}\right)$$

and hence

$$\lambda_{1,2} = \frac{-\frac{Ck_2}{m\bar{z}^2} \pm \sqrt{\left(\frac{Ck_2}{m\bar{z}^2}\right)^2 - 4\left(\frac{Ck_1}{m\bar{z}^2} - \frac{2g}{m}\right)}}{2}$$

[2 points]. This leads to the following conditions for asymptotic stability (i.e. all eigenvalues have real part < 0):

- $k_2 > 0$ **[1 point]**
- $k_1 \geq \frac{2g\bar{z}^2}{C}$ **[1 point]**

7. No, since we saw in the previous task that $k_2 > 0$ is required for asymptotic stability **[1 point]**.

Exercise 4

1	2	3	4	Exercise
3	9	8	5	25 Points

1. The system is linear (0.5p) and time invariant (0.5p). For $u(t) = 0$, (**??**) is a scalar linear system with eigenvalue $\lambda = a$. (0.5p) Therefore, it is asymptotically stable if $a < 0$, stable if $a = 0$ and unstable if $a > 0$. (1.5p)

2. (a) (1p)

$$\begin{bmatrix} \dot{z}(t) \\ \dot{k}(t) \end{bmatrix} = \begin{bmatrix} (a - k(t))z(t) \\ z^2(t) \end{bmatrix} \tag{1}$$

   (b) The extended system (**??**) is not linear, because for example we have the term $z^2(t)$, but is time-invariant. (2p)

   (c) The equilibrium points of system (**??**) are all the points in the set $E := \{(z, k) \mid z = 0\}$. (1p)

   (d) The linearized matrix computed at the equilibrium points is

$$\begin{bmatrix} a - k & -z \\ 2z & 0 \end{bmatrix}_{(0,b)} = \begin{bmatrix} a - b & 0 \\ 0 & 0 \end{bmatrix},$$

   whose eigenvalues are $\Lambda = \{0, a - b\}$. (2p) Therefore, if $b < a$ the equilibrium is unstable, if $b \geq a$ the linearization technique is inconclusive. (3p)

3. (a) The Lie derivative of $V_b(z, k)$ according to the dynamics given in system (**??**) is (2p)

$$\dot{V}_b(z, k) = z\dot{z} + (k - b)\dot{k} = z^2(a - k) + (k - b)z^2 = z^2(a - b).$$

   (b) Consider the equilibrium point $\hat{x} = (0\ b)^\top$ and let S be an open ball centered at $\hat{x}$ with radius ε.(0.5p) Notice that $V_b(x) > 0$ for all $x \in S \backslash \{\hat{x}\}$ (0.5p)and $V_b(\hat{x}) = 0$ (0.5p).

   - For $b > a$, according to the previous subtask $\dot{V}_b(x) \leq 0$ for all $x \in S$, hence the equilibrium $\hat{x}$ is stable.(1.5p)
   - For $b = a$, according to the previous subtask $\dot{V}_b(x) = 0$ for all $x \in S$, hence the equilibrium $\hat{x}$ is stable.(1.5p)
   - For $b < a$, according to linearization method studied above, the equilibrium $\hat{x}$ is unstable.(1.5p)

4. (a) First of all note that the value of $k(t)$ is monotonically non-decreasing (0.5p). We are going to prove that $z(t) \to 0$ (1p).
   Consider the trajectory generated by an arbitrary initial point (z_0, k_0). There are three possible cases.

   - $z_0 = 0$, then (z_0, k_0) is an equilibrium point and $z(t) = 0$ for all $t > 0$.

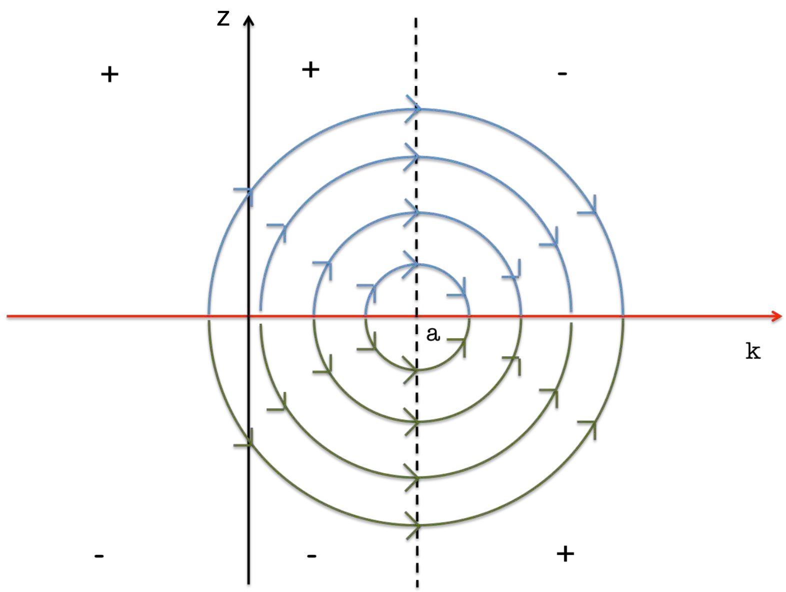

Figure 1: Trajectories of system (**??**). In red are the equilibrium points.

- $z_0 \neq 0$ and $k_0 > a$, then $k(t) > a$ for all $t > 0$. Consequently, $(a - k(t)) < 0$ for all $t > 0$ and therefore $z(t) \to 0$. (0.5p)
- $z_0 \neq 0$ and $k_0 \leq a$, then both $k(t)$ and $|z(t)|$ increases until a time $\bar{t} > 0$ when $k(\bar{t}) > a$. Since the system is time-invariant we can analyze what happens for $t > \bar{t}$ by considering what happens to a trajectory starting from time $t = 0$ in $(z_0, k_0) = (z(\bar{t}), k(\bar{t}))$. From the previous case, we can therefore conclude that $z(t) \to 0$. (0.5p)

(b) All the points in the axis $z = 0$ are equilibrium points hence trajectories starting from there remain there. (0.5p) Along the axis $k = a$, $\dot{z} = 0$ therefore the trajectories must be perpendicular to this axis. Moreover the sign of $\dot{z}$ is as given in Figure 1. Finally, $\dot{V}_a(z, k) = 0$ for all (z, k). Which means that $z^2 + (k - a)^2 = const$. In other words the trajectories are semicircles around the point $(z, k) = (0, a)$ (1p), with the direction given by the sign of $\dot{z}$. (1p)