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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	Exercise
4	5	4	7	5	25 Points

1. A is triangular, hence the eigenvalues are $\lambda_1 = \alpha$ and $\lambda_2 = \alpha^2$. If $\alpha \in \{0, 1\}$, then $\alpha = \alpha^2$. In this case A is in Jordan canonical form, not diagonalizable and there exists only one eigenvector $v_1 = [1\ 0]^\top$. If $\alpha \notin \{0, 1\}$, the eigenvalues of A are distinct and A is diagonalizable. The eigenvectors in this case are

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 1 \\ \alpha^2 - \alpha \end{bmatrix}.$$

2. Let $\alpha < 0$. Then A is diagonalizable with

$$A = W\Lambda W^{-1}, \text{ where } W = \begin{bmatrix} 1 & 1 \\ 0 & \alpha^2 - \alpha \end{bmatrix} \text{ and } W^{-1} = \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} \alpha^2 - \alpha & -1 \\ 0 & 1 \end{bmatrix}.$$

We obtain

$$\begin{aligned} e^{At} = We^{\Lambda t}W^{-1} &= \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} 1 & 1 \\ 0 & \alpha^2 - \alpha \end{bmatrix} \begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{\alpha^2 t} \end{bmatrix} \begin{bmatrix} \alpha^2 - \alpha & -1 \\ 0 & 1 \end{bmatrix} = \\ &= \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} e^{\alpha t} & e^{\alpha^2 t} \\ 0 & (\alpha^2 - \alpha)e^{\alpha^2 t} \end{bmatrix} \begin{bmatrix} \alpha^2 - \alpha & -1 \\ 0 & 1 \end{bmatrix} = \\ &= \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} (\alpha^2 - \alpha)e^{\alpha t} & e^{\alpha^2 t} - e^{\alpha t} \\ 0 & (\alpha^2 - \alpha)e^{\alpha^2 t} \end{bmatrix}. \end{aligned}$$

Alternative solution:

To show that the given matrix is indeed the matrix exponential e^{At} we can show that it fulfills the differential equation $\frac{d}{dt}e^{At} = Ae^{At}$ and that $e^{A0} = I$. The latter statement clearly holds. For the former, differentiating the given matrix exponential we obtain

$$\frac{d}{dt}e^{At} = \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} \alpha(\alpha^2 - \alpha)e^{\alpha t} & \alpha^2 e^{\alpha^2 t} - \alpha e^{\alpha t} \\ 0 & \alpha^2(\alpha^2 - \alpha)e^{\alpha^2 t} \end{bmatrix}.$$

Further, we find that

$$\begin{aligned} Ae^{At} &= \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} \alpha(\alpha^2 - \alpha)e^{\alpha t} & \alpha e^{\alpha^2 t} - \alpha e^{\alpha t} + (\alpha^2 - \alpha)e^{\alpha^2 t} \\ 0 & \alpha^2(\alpha^2 - \alpha)e^{\alpha^2 t} \end{bmatrix} \\ &= \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} \alpha(\alpha^2 - \alpha)e^{\alpha t} & \alpha^2 e^{\alpha^2 t} - \alpha e^{\alpha t} \\ 0 & \alpha^2(\alpha^2 - \alpha)e^{\alpha^2 t} \end{bmatrix}. \end{aligned}$$

For $\alpha = 0$, A is not diagonalizable but it is nilpotent and we find

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

3. Since $\alpha^2 > 0 \ \forall \ \alpha \neq 0$ it is immediately clear that the system cannot be stable for $\alpha \neq 0$. For $\alpha = 0$ we can see from the matrix exponential that the system is also not stable. Hence, the system is unstable for all α.

4. It is easy to see that for $\alpha = 0$ the system is controllable, hence such an input must exist. To find one we can, for instance, use the minimum energy input which we can compute with the controllability gramian

$$\begin{aligned} W_C(1) &= \int_0^1 e^{At} B B^\top e^{A^\top t} dt = \\ &= \int_0^1 \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} dt = \\ &= \int_0^1 \begin{bmatrix} t^2 & t \\ t & 1 \end{bmatrix} dt = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}. \end{aligned}$$

With this we obtain

$$\begin{aligned} u(t) &= B^\top e^{A^\top (1-t)} W_C(1)^{-1} x(1) = \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1-t & 1 \end{bmatrix} 12 \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 12 \begin{bmatrix} 1-t & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} = 12 \left(\frac{1}{2} - t \right) = 6 - 12t. \end{aligned}$$

5. For $\alpha = -1$, B contains only zeros and the input does not influence the system at all. The state transition is therefore given by $e^{At} x_0$. Using the results from subquestion 2 we obtain

$$x(t) = e^{At} x_0 = \frac{1}{2} \begin{bmatrix} 2e^{-t} & e^t - e^{-t} \\ 0 & 2e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}.$$

Hence, at $t = 1$ the desired state will be reached for any input $u(t)$.

Exercise 2

1	2	3	4	Exercise
9	6	4	6	25 Points

1. - [**3 points**] Mechanical system: $m \cdot \ddot{x}(t) = \kappa \cdot I(t) - f \cdot x(t) - d \cdot v(t)$
   - [**3 points**] Electrical system: $L \cdot \frac{d}{dt} I(t) = -R \cdot I(t) - \kappa \cdot v(t) + u(t)$
   - Faraday's law: $(F(t) = B \cdot n \cdot D \cdot \pi \cdot I(t) = \kappa \cdot I(t))$
   - Lorentz' law: $(U_{ind}(t) = B \cdot n \cdot D \cdot \pi \cdot v(t) = \kappa \cdot v(t))$

   With state vector $z = [I(t), x(t), v(t)]^T$, $u(t) = F(t)$ and $y(t) = x(t)$, the system matrices read:

   - [**2 points**] $A = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{\kappa}{L} \\ 0 & 0 & 1 \\ \frac{\kappa}{m} & -\frac{f}{m} & -\frac{d}{m} \end{bmatrix}$
   - [**1 point**] $B = \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix}$
   - $C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

2. With $L = m = f = d = \kappa = 1$ and $R = 2$, the state space matrices read
   $A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

   Stability: [**2 points**]

$$\begin{aligned} \det(sI - A) &= (s+2)\left(s(s+1)+1\right) + s && (1) \\ &= (s+2)(s^2+s+1) + s && (2) \\ &= s^3 + 3s^2 + 4s + 2 && (3) \\ &= (s+1)(s^2+2s+2) && (4) \end{aligned}$$

   Use the hint for the last step.

$$\begin{aligned} s_1 &= -1 && (5) \\ s_{2,3} &= \frac{-2 \pm \sqrt{4-8}}{2} && (6) \\ &= -1 \pm i && (7) \end{aligned}$$

   All $\mathrm{Re}[s_i] \leq 0$, i.e., the system is stable.

   Controllability: [**2 points**] P:= $\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix}$. Since $\det(P) \neq 0$, P has full rank, and therefore the system is controllable.

Observability: [**2 points**] O:= $\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$. Since $\det(O) \neq 0$, O has full rank, and therefore the system is observable.

3. [**4 points**] The transfer function is defined as $G(s) = C(sI - A)^{-1}B + D$. Because $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, we only need the (first column, second row)-element of $(sI - A)^{-1}$. Consequently, $G(s) = \frac{fg-di}{det(A)} = \frac{1}{(s+1)(s+1-i)(s+1+i)}$ (with f, g, d, i being some of the original elements of $sI - A$).

4. To derive an expression for $y(t)$ we need to perform an inverse Laplace transform, hence we expand the transfer function in partial fractions.

$$\frac{1}{(s+1)(s+1-i)(s+1+i)} = \frac{A}{s+1} + \frac{B}{s+1-i} + \frac{C}{s+1+i} \tag{8}$$

$$= \frac{A(s^2+2s+2) + B(s^2+(s+i)s+1+i) + C(s^2+(2-i)s+1-i)}{(s+1)(s+1-i)(s+1+i)} \tag{9}$$

$$= \frac{(A+B+C)s^2 + (2A+(2+i)B+(2-i)C)s + (2A+(1+i)B+(1-i)C)}{(s+1)(s+1-i)(s+1+i)} \tag{10}$$

By comparison of coefficients, we obtain A=1, B=C=-0.5 [**3 points**]. Hence, the Laplace transform of the output is

$$Y(s) = \frac{1}{s+1} + \frac{-0.5}{s+1-i} + \frac{-0.5}{s+1+i}. \tag{11}$$

The inverse Laplace transform is [**3 points**]

$$y(t) = e^{-t} - 0.5e^{(-1+i)t} - 0.5e^{(-1-i)t} \tag{12}$$

$$= e^{-t} - e^{-t}\left(\frac{e^{it}+e^{-it}}{2}\right) \tag{13}$$

$$= e^{-t} - e^{-t}\cos(t) \tag{14}$$

$$= e^{-t}(1-\cos(t)). \tag{15}$$

Alternative derivation:

$$\frac{1}{(s+1)(s+1-i)(s+1+i)} = \frac{A}{s+1} + \frac{Bs+C}{(s+1)^2+1} \tag{16}$$

$$= \frac{A(s^2+2s+2) + B(s^2+s) + C(s+1)}{(s+1)\left((s+1)^2+1\right)} \tag{17}$$

$$= \frac{(A+B)s^2 + (2A+B+C)s + (2A+C)}{(s+1)\left((s+1)^2+1\right)} \tag{18}$$

By comparison of coefficients, we obtain A=1, B=C=-1 [**3 points**]. Hence, the Laplace transform of the output is

$$Y(s) = \frac{1}{s+1} - \frac{s+1}{(s+1)^2+1}. \tag{19}$$

The inverse Laplace transform is [**3 points**]

$$\begin{aligned} y(t) &= e^{-t} - e^{-t}\cos(t) &(20)\\ &= e^{-t}(1-\cos(t)). &(21) \end{aligned}$$

Exercise 3

1	2	3	4	Exercise
6	6	6	7	25 Points

1. The equilibrium points are the solutions of

$$0 = (x^2 + y^2 - 1)x - y, \tag{22}$$
$$0 = (x^2 + y^2 - 1)y + x. \tag{23}$$

It is immediate to see that the point $(x, y) = (0, 0)$ satisfies (22) and (23), therefore the origin is the unique equilibrium point of the system. The linearization of the system around the origin is

$$A = \begin{bmatrix} \frac{\partial\left[(x^2+y^2-1)x-y\right]}{\partial x} & \frac{\partial\left[(x^2+y^2-1)x-y\right]}{\partial y} \\ \frac{\partial\left[(x^2+y^2-1)y+x\right]}{\partial x} & \frac{\partial\left[(x^2+y^2-1)y+x\right]}{\partial y} \end{bmatrix}_{(0,0)} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}. \tag{24}$$

The eigenvalues are $\lambda_{1,2} = -1 \pm \sqrt{-1}$. Since $Re(\lambda_{1,2}) < 0$ we can conclude, from Theorem 7.1, that the origin is locally asymptotically stable.

2. Note that $r^2(t) = x^2(t) + y^2(t)$, hence

$$\frac{d}{dt}r^2(t) = \frac{d}{dt}(x^2(t) + y^2(t))$$
$$2r(t)\dot{r}(t) = 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)$$
$$2r(t)\dot{r}(t) = 2x(t)\left[(x^2(t) + y^2(t) - 1)x(t) - y(t)\right] + 2y(t)\left[(x^2(t) + y^2(t) - 1)y(t) + x(t)\right]$$
$$2r(t)\dot{r}(t) = 2x^2(t)(r^2(t) - 1) - 2x(t)y(t) + 2y^2(t)(x^2(t) + y^2(t) - 1) + 2x(t)y(t)$$
$$2r(t)\dot{r}(t) = 2r^2(t)(r^2(t) - 1).$$

Therefore $\dot{r}(t) = r(t)(r^2(t) - 1)$.

To compute $\dot{\theta}(t)$ we can differentiate both sides of $y(t) = r(t)\sin(\theta(t))$.

$$\begin{aligned} \frac{d}{dt}y(t) &= \frac{d}{dt}(r(t)\sin(\theta(t))) \\ \dot{y}(t) &= \dot{r}(t)\sin(\theta(t)) + r(t)\cos(\theta(t))\dot{\theta}(t) \\ (x^2(t) + y^2(t) - 1)y(t) + x(t) &= r(t)(r^2(t) - 1)\sin(\theta(t)) + r(t)\cos(\theta(t))\dot{\theta}(t) \\ (r^2(t) - 1)r(t)\sin(\theta(t)) + r(t)\cos(\theta(t)) &= r(t)(r^2(t) - 1)\sin(\theta(t)) + r(t)\cos(\theta(t))\dot{\theta}(t) \\ r(t)\cos(\theta(t)) &= r(t)\cos(\theta(t))\dot{\theta}(t) \\ 1 &= \dot{\theta}(t). \end{aligned}$$

Hence the system in polar coordinates is

$$\dot{r}(t) = r(t)(r^2(t) - 1), \tag{25}$$
$$\dot{\theta}(t) = 1. \tag{26}$$

3. Note that equations (25) and (26) are decoupled. The angle $\theta(t)$ is constantly increasing over time irrespective of the value of $r(t)$. On the other hand, if $r(t) < 1$ then $\dot{r}(t) < 0$, if $r(t) > 1$ then $\dot{r}(t) > 0$ and $\dot{r}(t) = 0$ if $r(t) = 1$. Therefore if $r(t) < 1$ it will decrease towards zero, if $r(t) > 1$ it will increase towards infinity and if $r(t) = 1$ it will stay constant. Therefore the trajectories are as illustrated in Figure 1.

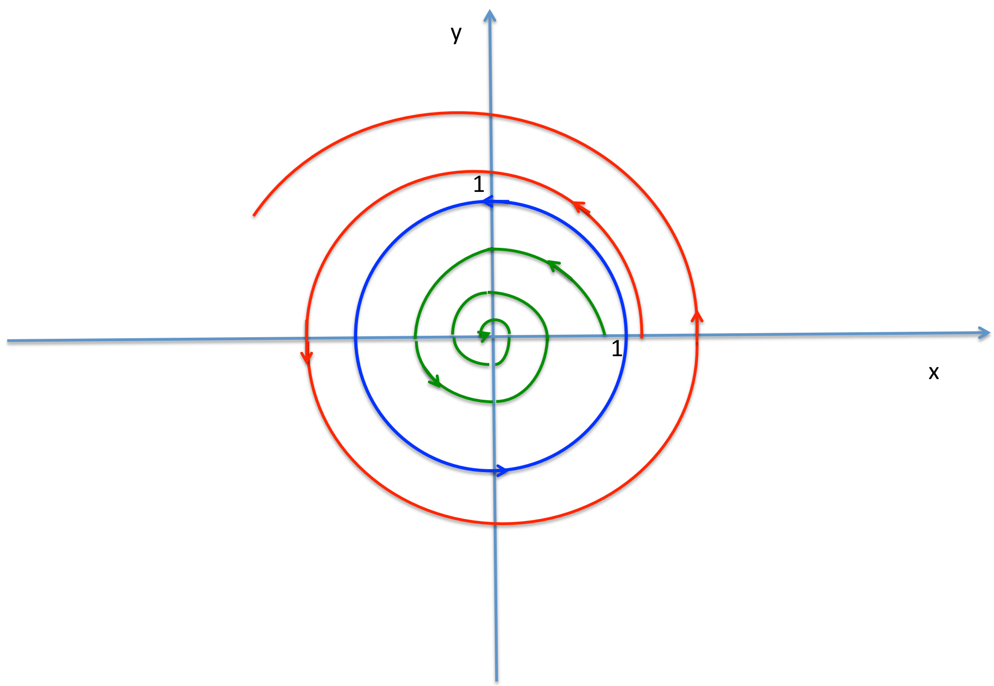

Figure 1: Three sample trajectories. Green: $r(0) < 1$. Blue: $r(0) = 1$. Red: $r(0) > 1$.

From this analysis we can conclude that the circle of radius one is a periodic orbit. Moreover, the equilibrium point in the origin is a trivial periodic orbit.

4. From the analysis of part 3 it is clear that any circle C_ρ of radius $\rho \leq 1$ is an invariant set. In fact r is non-increasing for any value of $r \leq 1$ therefore any trajectory starting inside a set C_ρ of radius $\rho \leq 1$ cannot leave it. The sets C_ρ are the level sets of the function

$$V(x, y) = x^2 + y^2.$$

To prove that this is a local Lyapunov function for the origin we need to prove that there exists an open set S containing the origin such that:

(a) $V(x, y) > 0$ for any $(x, y) \in S \neq (0, 0)$;

(b) $\dot{V}(x, y) < 0$ for any $(x, y) \in S \neq (0, 0)$.

To this end, we can use the suggested set $S = \{(x, y) | x^2 + y^2 < 1\}$. Condition (a) is always true. The time derivative of $V(x(t), y(t))$ is

$$\dot{V}(x(t), y(t)) = 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t) = 2(x^2(t) + y^2(t))(x^2(t) + y^2(t) - 1) = 2r^2(t)(r^2(t) - 1),$$

which is strictly negative on $S \backslash (0, 0)$. Therefore also Condition (b) holds and $V(x, y)$ is a local Lyapunov function. The asymptotic stability of the origin follows from Theorem 7.3.
Finally, we can conclude that S is contained in the domain of attraction of the origin. Actually, from the analysis done in part 3, we can conclude that S is the entire domain of attraction.

Exercise 4

1	2	3	4	Exercise
7	6	7	5	25 Points

1. The output after k time-steps is given by

$$\begin{aligned} y(k) &= Cx(k) + Du(k) \\ &= C\left(A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1}Bu(i)\right) + Du(k) \\ &= CA^k x(0) + \sum_{i=0}^{k-1} CA^{k-i-1}Bu(i) + Du(k) \end{aligned}$$

[1 point for providing this formula]

Stacking the outputs for $k = 0, \dots, N-1$ to make use of U and Y, this can be written in terms of a large system of linear equations as follows:

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}}_{Q} x(0) + \underbrace{\begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \dots & D \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix}}_{U}$$

[3 points for stacking the vectors correctly]

Collecting all known quantities on the right-hand side, we can rewrite this as

$$Qx(0) = Y - HU, \tag{27}$$

which can be solved for the only unknown $x(0)$ under some assumptions (see next part). **[1 point for writing the system of linear equations in the required form]**

In the notation of the exercise, $M = Q \in \mathbb{R}^{pN \times n}$ and $q = Y - HU \in \mathbb{R}^{pN}$. **[2 points for the correct dimensions]**

2. Because $N = n$, the coefficient matrix of the system of equations (27) is exactly the observability matrix Q, i.e. we have to solve (27) to find $x(0)$. **[2 points for noticing that $M = Q$]**

Since $p = 1$, it follows that Q is square. Therefore (27) has a unique solution if and only if Q has full rank. Equivalently (for square matrices), $\det(Q) \neq 0$, Q^{-1} has to exist, or $\mathcal{R}(Q) = \mathbb{R}^n$. **[3 points for a unique solution condition + 1 point for making the connection to $\text{rank}(Q) = n$]**

3. Using the given values in (27) leads to the system of linear equations [**3 points for the correct system**]

$$\begin{bmatrix} 1 & 0.9 \\ 2.8 & 2.4 \end{bmatrix} x(0) = \underbrace{\begin{bmatrix} 2.9 \\ 9.9 \end{bmatrix}}_{Y} - \underbrace{\begin{bmatrix} 0 & 0 \\ 1.9 & 0 \end{bmatrix}}_{\mathcal{M}} \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{U} = \begin{bmatrix} 2.9 \\ 8 \end{bmatrix},$$

which is solved by

$$x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

[**4 points for the correct solution**]

4. In the presence of measurement noise, it might be impossible to reliably reconstruct the initial state. [**1 point**]

   In order to get a good estimate of the system state, it is therefore better to design a state estimator such as a Luenberger Observer or Kalman Filter. [**2 points**]

   In order for such estimation methods to work, the system has to be detectable. Note that this is a slightly weaker condition than observability, i.e. detectable systems can be unobservable but an observer will still work. [**1 point**]

   When the state estimator for a detectable system is properly designed, the estimation error will tend to zero, providing a very good estimate of the system state in the long run. [**1 point**]