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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	Exercise
4	7	7	7	25 Points

1. The eigenvalues of the system are $\Lambda(A) = \{-1, 6\}$. Since $|6| > 1$, the system is unstable hence it is not asymptotically stable.

2. The transfer function is

$$G(z) = C(zI - A)^{-1}B = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} z-1 & -5 \\ -2 & z-4 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 2 \end{bmatrix}$$
$$= \frac{1}{(z-1)(z-4)-10} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} z-4 & 5 \\ 2 & z-1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 2 \end{bmatrix} = \frac{4.5z+6}{(z+1)(z-6)}.$$

There are no zero/pole cancellations hence the system is both observable and controllable. Indeed non-observable or non-controllable modes do not appear in the transfer function and this is possible only if zero/pole cancellations are present.

3. Substituting $u(k) = Kx(k)$ in system (**??**) yields

$$x(k+1) = (A + BK)x(k) = A_K x(k),$$
$$y(k) = Cx(k).$$

hence $A_K = (A + BK)$. The closed loop system is asymptotically stable if and only if all the eigenvalues of A_K are inside the unit circle.

$$A_K = (A + BK) = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 + 0.5k_1 & 5 + 0.5k_2 \\ 2 + 2k_1 & 4 + 2k_2 \end{bmatrix}.$$

As the hint suggested, select $k_1 = -1$ so that A_K is upper triangular

$$A_K = \begin{bmatrix} 0.5 & 5 + 0.5k_2 \\ 0 & 4 + 2k_2 \end{bmatrix},$$

and note that the eigenvalues are the elements on the diagonal. Therefore any k_2 such that $|4 + 2k_2| < 1$ leads to a stable matrix, choose for example $k_2 = -2$ so that

$$A_K = \begin{bmatrix} 0.5 & 4 \\ 0 & 0 \end{bmatrix}.$$

4. The error dynamic is

$$e(k+1) = \hat{x}(k) - x(k) = A\hat{x}(k) + Bu(k) + L\left[y(k) - C\hat{x}(k)\right] - Ax(k) - Bu(k)$$
$$= \left[A - LC\right] e(k) = A_L e(k).$$

Note that the corresponding solution is

$$e(k) = A_L^k e(0),$$

and that, since $\hat{x}(0)$ is arbitrary, also $e(0)$ is arbitrary. Therefore the error goes to zero in $\bar{k}$ steps if and only if the matrix $A_L^{\bar{k}}$ is the zero matrix. That is A_L is nilpotent, i.e. all its eigenvalues are zero.

$$\begin{aligned} A_L = A - LC &= \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 - l_1 & 5 - 2l_1 \\ 2 - l_2 & 4 - 2l_2 \end{bmatrix}. \end{aligned}$$

Selecting $l_1 = 1$ and $l_2 = 2$ yields

$$A_L = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix},$$

which is nilpotent as required, with $\bar{k} = 2$.

Exercise 2

1	2	3	4	Exercise
5	6	7	7	25 Points

1. To check stability we compute the eigenvalues of the system matrix A.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda+1 & 0 & 0 \\ -1 & \lambda-3 & 5 \\ -1 & 0 & \lambda+2 \end{bmatrix} = (\lambda+1)(\lambda-3)(\lambda+2).$$

   Hence, the eigenvalues are $\lambda_1 = -1, \lambda_2 = 3, \lambda_3 = -2$. λ_2 is real and positive and the system is therefore unstable.

2. With $\widehat{x}(t) = Tx(t)$ and $x(t) = T^{-1}\widehat{x}(t)$ it holds that

$$\begin{aligned} \dot{\widehat{x}}(t) &= T\dot{x}(t) = TAx(t) + TBu(t) = TAT^{-1}\widehat{x}(t) + TBu(t) \\ y(t) &= Cx(t) = CT^{-1}\widehat{x}(t). \end{aligned}$$

   Hence, the matrices $\widehat{A}, \widehat{B}, \widehat{C}$ can be found as follows:

$$\begin{aligned} \widehat{A} &= TAT^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\ \widehat{B} &= TB = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^\top \\ \widehat{C} &= CT^{-1} = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \end{aligned}$$

   Clearly, $\widehat{A}$ is diagonal. Consequently, T^{-1} has as columns the eigenvectors of A.

3. To check controllability of the new system we compute the controllability matrix:

$$\widehat{P} = \begin{bmatrix} \widehat{B} & \widehat{A}\widehat{B} & \widehat{A}^2\widehat{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -3 & -9 \\ 0 & 0 & 0 \end{bmatrix}.$$

   $\widehat{P}$ does not have full rank since the third row consists only of zeros and hence the system is not controllable.

   To check observability of the new system we compute the observability matrix:

$$\widehat{O} = \begin{bmatrix} \widehat{C} \\ \widehat{C}\widehat{A} \\ \widehat{C}\widehat{A}^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & -2 \\ 2 & 0 & 4 \end{bmatrix}.$$

   $\widehat{O}$ does not have full rank since the second column consists only of zeros and hence the system is not observable.
   These results are also clear because $\widehat{A}$ is a diagonal matrix and both $\widehat{B}$ and $\widehat{C}$ contain a zero entry.

To address stabilizability and detectability we first note that the new system (2) is the Kalman decomposition of the original system (1) with $\widehat{x}_1$ as the part which is controllable and observable, $\widehat{x}_2$ as the part which is controllable but not observable, $\widehat{x}_3$ as the part which is not controllable but observable and no part which is neither controllable nor observable. Writing $\widehat{A}$ in accordance with the Kalman decomposition form as

$$\widehat{A} = \begin{bmatrix} \widehat{A}_{11} & 0 & 0 \\ 0 & \widehat{A}_{22} & 0 \\ 0 & 0 & \widehat{A}_{33} \end{bmatrix}$$

we can conclude that the system is stabilizable because $\widehat{A}_{33} = -2$ is real and negative (the uncontrollable part $\widehat{x}_3$ is stable) but not detectable because $\widehat{A}_{22} = 3$ is real and positive (the unobservable part $\widehat{x}_2$ is unstable).

Since the matrix T is an invertible transformation, the two systems are equivalent and hence the original system is controllable/observable/stabilizable/detectable if and only if the new system is controllable/observable/stabilizable/detectable. Hence, the original system is neither controllable nor observable nor detectable, but it is stabilizable.

4. To compute the zero input response we make use of the results of subquestion (2), from which we know that $A = T^{-1}\widehat{A}T$ and $CT^{-1} = \widehat{C}$.

$$\begin{aligned} y(t) &= Ce^{At}x_0 = CT^{-1}e^{\widehat{A}t}Tx_0 = \widehat{C}e^{\widehat{A}t}Tx_0 \\ &= \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} Tx_0 \\ &= \begin{bmatrix} 2e^{-t} & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} x_0 \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} & 0 & e^{-2t} \end{bmatrix} x_0 \end{aligned}$$

Clearly $y(t) \to 0$ as $t \to \infty$ which tells us that the output converges to zero even though the system is unstable as seen in part 1 of this exercise. This is a consequence of the system not being detectable as found in part 3 of this exercise which tells us that there are unstable parts which are unobservable.

Exercise 3

1	2	3	4	Exercise
7	7	7	4	25 Points

1. By applying Newton's law to the first mass we get

$$m\ddot{\xi}_1 = -k\xi_1 - d\dot{\xi}_1 + k(\xi_2 - \xi_1). \quad \textbf{[2 Points]}$$

Similarly, the equation of motion for the second mass is

$$m\ddot{\xi}_2 = -k(\xi_2 - \xi_1) - d\dot{\xi}_2 + k(u - \xi_2). \quad \textbf{[2 Points]}$$

Hence,

$$\begin{aligned}\ddot{\xi}_1 &= -2\frac{k}{m}\xi_1 - \frac{d}{m}\dot{\xi}_1 + \frac{k}{m}\xi_2\\ \ddot{\xi}_2 &= -2\frac{k}{m}\xi_2 - \frac{d}{m}\dot{\xi}_2 + \frac{k}{m}\xi_1 + \frac{k}{m}u,\end{aligned}$$

which in state space form is given by

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}\xi_1\\ \dot{\xi}_1\\ \xi_2\\ \dot{\xi}_2\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ -2\frac{k}{m} & -\frac{d}{m} & \frac{k}{m} & 0\\ 0 & 0 & 0 & 1\\ \frac{k}{m} & 0 & -2\frac{k}{m} & -\frac{d}{m}\end{pmatrix}\begin{pmatrix}\xi_1\\ \dot{\xi}_1\\ \xi_2\\ \dot{\xi}_2\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 0\\ \frac{k}{m}\end{pmatrix}u. \quad \textbf{[2 Points]}$$

The output can be written as

$$y = \begin{pmatrix}1 & 0 & -1 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \dot{\xi}_1\\ \xi_2\\ \dot{\xi}_2\end{pmatrix}. \quad \textbf{[1 Point]}$$

2. Using the given change of coordinates

$$\begin{aligned}\dot{\hat{x}}_1 &= \frac{1}{2}(\dot{\xi}_1(t) + \dot{\xi}_2(t)) = \hat{x}_2, \quad \textbf{[0.5 Points]}\\ \dot{\hat{x}}_2 &= \frac{1}{2}(\ddot{\xi}_1 + \ddot{\xi}_2)\\ &= \frac{1}{2}\Big(- 2\frac{k}{m}\xi_1 - \frac{d}{m}\dot{\xi}_1 + \frac{k}{m}\xi_2 - 2\frac{k}{m}\xi_2 - \frac{d}{m}\dot{\xi}_2 + \frac{k}{m}\xi_1 + \frac{k}{m}u\Big)\\ &= -\frac{k}{m}z_1 - \frac{d}{m}\dot{z}_1 + \frac{k}{2m}u, \quad \textbf{[2 Points]}\end{aligned}$$

$$\begin{aligned}\dot{\hat{x}}_3 &= \frac{1}{2}(\dot{\xi}_1(t) - \dot{\xi}_2(t)) = \hat{x}_4, \quad \textbf{[0.5 Points]}\\ \dot{\hat{x}}_4 &= \frac{1}{2}(\ddot{\xi}_1 - \ddot{\xi}_2)\\ &= \frac{1}{2}\Big(- 2\frac{k}{m}\xi_1 - \frac{d}{m}\dot{\xi}_1 + \frac{k}{m}\xi_2 + 2\frac{k}{m}\xi_2 + \frac{d}{m}\dot{\xi}_2 - \frac{k}{m}\xi_1 - \frac{k}{m}u\Big)\\ &= -3\frac{k}{m}z_2 - \frac{d}{m}\dot{z}_2 - \frac{k}{2m}u. \quad \textbf{[2 Points]}\end{aligned}$$

One can see that this change of coordinates decouples the the equations of motion and has the matrix form representation

$$\frac{d}{dt}\hat{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{d}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{3k}{m} & -\frac{d}{m} \end{pmatrix} \hat{x}(t) + \begin{pmatrix} 0 \\ \frac{k}{m} \\ 0 \\ -\frac{k}{m} \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 0 & 0 & 2 & 0 \end{pmatrix} \hat{x}(t). \quad \textbf{[1 Point]}$$

Hence, $\alpha_1 = \frac{k}{m}, \alpha_2 = \frac{3k}{m}, \beta_1 = \beta_2 = \frac{d}{m}, \gamma_1 = \gamma_2 = \frac{k}{m}$ and $\delta = 2$. **[1 Point]**

3. Define the state variables $x = (x_1, x_2, x_3, x_4)^\top := (z_1, \dot{z}_1, z_2, \dot{z}_2)^\top$, which gives

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\alpha_1 & -\beta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\alpha_2 & -\beta_2 \end{pmatrix}}_{=:A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma_1 \\ 0 \\ \gamma_2 \end{pmatrix} u,$$

where $\alpha_1 = \frac{k}{m}$, $\alpha_2 = \frac{3k}{m}$, $\beta_1 = \beta_2 = \frac{d}{m}$, $\gamma_1 = \frac{k}{2m}$, and $\gamma_2 = \frac{-k}{2m}$. **[2 Points]**

Note that A is block diagonal with $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where

$$A_i = \begin{pmatrix} 0 & 1 \\ -\alpha_i & -\beta_i \end{pmatrix} \quad \text{for } i = 1, 2.$$

Therefore,

$$\det(\lambda I_4 - A) = \det(\lambda I_2 - A_1)\det(\lambda I_2 - A_2). \quad \textbf{[3 Points]}$$

Furthermore,

$$\det(\lambda I_2 - A_i) = \det \begin{pmatrix} \lambda & -1 \\ \alpha_i & \lambda + \beta_i \end{pmatrix} = \lambda^2 + \beta_i \lambda + \alpha_i \stackrel{!}{=} 0$$

$\Rightarrow \lambda_{1,2}^{(i)} = \frac{-\beta_i \pm \sqrt{\beta_i^2 - 4\alpha_i}}{2}$ for $i = 1, 2$. Since we know that $\alpha_i, \beta_i > 0$ for $i = 1, 2$, all eigenvalues of A have strictly negative real part. Hence, the system is asymptotically stable. **[2 Points]**

4. In the new coordinates the output is given by $y(t) = 2z_2(t)$, i.e., $C = \begin{pmatrix} 0 & 0 & 2 & 0 \end{pmatrix}$. Since the state matrix is block diagonal $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, the observability matrix has the following form

$$Q = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix},$$

which clearly has not full rank. Hence the system is not observable. **[3 Points]**

A physical interpretation why the system is not observable is that given the measurement, which is the difference between position of the first mass and of the second, we cannot uniquely determine the absolute position of the masses (which are states). **[1 Point]**

Exercise 4

1	2	3	4	5	Exercise
4	6	6	6	3	25 Points

1. The system is only linear, if it admits homogeneity and additivity. Since the initial condition of the second experiment is twice the initial condition of the first experiment, also the system response should double. For the given responses we have

$$2\frac{1}{4e^t-3} = \frac{2}{4e^t-3} \neq \frac{2}{7e^t-6} \tag{1}$$

and hence, the system cannot be linear.

2. Setting $t = 0$ in the provided solution, one can easily verify that $x(0) = x_0$. Taking the derivative of the solution yields

$$\begin{aligned} \dot{x}(t) &= -\frac{x_0(e^t + 3e^t x_0)}{(e^t - 3x_0 + 3e^t x_0)^2} && (2)\\ &= -\underbrace{\frac{x_0(e^t + 3e^t x_0 - 3x_0)}{(e^t - 3x_0 + 3e^t x_0)^2}}_{x(t)} - 3\underbrace{\frac{x_0^2}{(e^t - 3x_0 + 3e^t x_0)^2}}_{x^2(t)} && (3)\\ &= -x(t) - 3x^2(t) && (4) \end{aligned}$$

3. The equilibria can be found be setting the system equation to zero:

$$0 = -2\hat{x}^2 - \hat{x} \tag{5}$$

Hence, we obtain

$$\hat{x}_1 = 0 \qquad \text{and} \quad \hat{x}_2 = -\frac{1}{3}. \tag{6}$$

Linearization around the equilibria gives

$$\dot{x}(t) = Ax(t) \tag{7}$$

with $A = -1 - 6\tilde{x}$. For $\tilde{x}_1$ we obtain $A = -1$ and consequently, the equilibrium is stable. For $\tilde{x}_2$ we find $A = 1$ which indicates instability.

4. The phase-plane plot is given in Fig. 1.

5. In order to show that a system is not time-invariant, we would have to measure the system response for a fixed initial condition two times in a row. Every time-invariant system (no matter if it is linear or not) must yield one and the same response for both measurements. Hence, if the both responses differ, the system cannot be time-invariant.

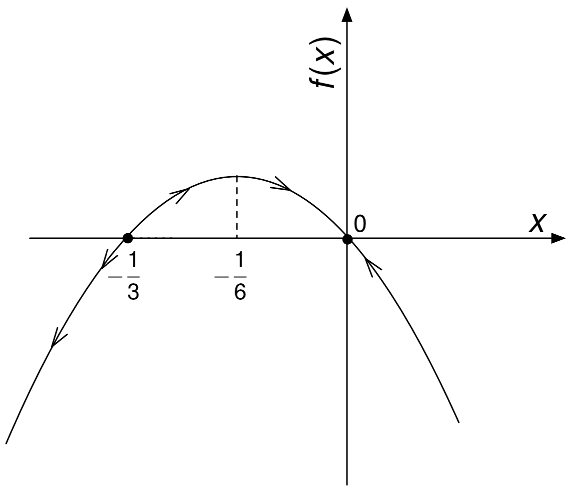

Figure 1: Phase-plane plot of the identified system.