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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	Exercise
10	6	9	25 Points

Consider the difference equation

$$y(k+2) + a_1 y(k+1) + a_0 y(k) = b_1 u(k+1) + b_0 u(k). \tag{1}$$

1. Consider the state $x_1(k) = y(k)$. This implies that $x_1(k+1) = y(k+1)$. Since (1) is time invariant we can shift it by one time-step and solve for $y(k+1)$. Therefore,

$$x_1(k+1) = -a_1 y(k) + b_1 u(k) - a_0 y(k-1) + b_0 u(k-1), \tag{2}$$

Set $x_2(k) = -a_0 y(k-1) + b_0 u(k-1)$. Equation (2) is then

$$x_1(k+1) = -a_1 x_1(k) + b_1 u(k) + x_2(k), \tag{3}$$

and

$$\begin{aligned} x_2(k+1) &= -a_0 y(k) + b_0 u(k), \\ &= -a_0 x_1(k) + b_0 u(k). \end{aligned} \tag{4}$$

Setting $x(k) = \begin{bmatrix} x_1(k) & x_2(k) \end{bmatrix}^T$, (3) and (4) lead to the following state space representation of the system[1]

$$\begin{aligned} x(k+1) &= \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k). \end{aligned} \tag{5}$$

The state space form matrices are $A = \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $D = 0$.

2. For $b_1 = 2b_0$, consider the controllability matrix

$$\begin{aligned} P = \begin{bmatrix} B & AB \end{bmatrix} &= \begin{bmatrix} b_1 & -a_1 b_1 + b_0 \\ b_0 & -a_0 b_1 \end{bmatrix} \\ &\begin{bmatrix} 2b_0 & (-2a_1+1)b_0 \\ b_0 & -2a_0 b_0 \end{bmatrix} \end{aligned} \tag{6}$$

The system is controllable if and only if P has full rank. The latter holds if the columns of P are linearly independent (equivalently if the determinant is not zero). To achieve this we need $b_0 \neq 0$ and $(2a_1 - 1)/(2a_0) \neq 2$. In other words, if $b_0 \neq 0$ and $a_1 - 2a_0 \neq 0.5$ the system is controllable.

The system is observable if and only if the observability matrix is full rank. The observability matrix is given by

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -a_1 & 1 \end{bmatrix}. \tag{7}$$

Q is always full rank, hence the system is observable for any values of a_0, a_1, b_0.

[1] Note that setting $x_2(k) = y(k+1) + a_1 y(k) - b_1 u(k)$ leads to the same state space representation.

3. The transfer function of the system is given by

$$\begin{aligned} G(z) &= C(zI - A)^{-1}B + D \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z + a_1 & -1 \\ a_0 & z \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} \\ &= \frac{1}{z^2 + a_1 z + a_0} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z & 1 \\ -a_0 & z + a_1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} \\ &= \frac{b_1 z + b_0}{z^2 + a_1 z + a_0}. \end{aligned} \tag{8}$$

The transfer function could also have been immediately deduced by observing that (5) is in the observable canonical form.

Consider now the case where $b_1 = 2b_0$. Pole-zero cancelations are related to the cases where controllability or observability is lost. Since the system is always observable, we need to examine the cases where it is uncontrollable, i.e. $b_0 = 0$ or $a_1 - 2a_0 = 0.5$. If $b_0 = 0$, no pole-zero cancelation occurs, but the transfer function is zero since matrix B of the state space form is also zero. If $a_1 - 2a_0 = 0.5$ the transfer function can be written as

$$\begin{aligned} G(z) &= \frac{2b_0(z + 0.5)}{(z + 2a_0)(z + 0.5)}, \\ &= \frac{2b_0}{(z + 2a_0)}. \end{aligned}$$

Therefore, if $a_1 - 2a_0 = 0.5$ we have one pole-zero cancelation.

Exercise 2

1	2	3	4	Exercise
5	8	7	5	25 Points

1. The system is linear, hence it is possible to write the system in state space form:
$$\dot{x}(t) = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$
$$y(t) = [1\ \ 0]\, x(t).$$
The system is unstable since it has eigenvalues -2 and 4.
The controllability matrix
$$P = [B\ \ AB] = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$
has rank 1; the system is not controllable.
The observability matrix
$$Q = [C\ \ CA]^T = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$$
has rank 2; the system is observable.

2. The closed loop system (with state matrix $A + BK$) has characteristic polynomial
$$\lambda^2 + (-k_1 - 2k_2 - 2)\lambda - 2k_1 - 4k_2 - 8.$$
The characteristic polynomial for a system with both eigenvalues at -4 is
$$\lambda^2 + 8\lambda + 16.$$
As the resulting set of linear equations has no solution, it is impossible to set both eigenvalues to -4. This can be expected given the controllability result in Part (1), although, in this case, the system can be stabilized.

3. The error dynamics are given by the ordinary differential equation
$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}} = [A - LC]\, e(t).$$
The error system (with state matrix $A - LC$) has characteristic polynomial
$$\lambda^2 + (l_1 - 2)\lambda - 4l_1 + 3l_2 - 8.$$
The characteristic polynomial for a system with eigenvalues at -1 and -1 is
$$\lambda^2 + 2\lambda + 1.$$
It follows that the estimator gain matrix $L = [4\ \ \frac{25}{3}]^T$ results in an error system with eigenvalues at -1 and -1.

4. From the second linear equality at equilibrium we have that $2\hat{x}_2 = -\hat{u}$. Plugging this value for $\hat{u}$ into the equilibrium equality of the first equation results in the relationship
$$2\hat{x}_1 = \hat{x}_2.$$

Exercise 3

1	2	3	Exercise
10	7	8	25 Points

1. From Kirchoff's laws and element equations:

$$V_s = R_1 i_L + L\frac{di_L}{dt} \implies \dot{x_1} = -\frac{R_1}{L}x_1 + \frac{u}{L}$$

$$C\frac{dV_c}{dt} = i_L \implies \dot{x_2} = \frac{1}{C}x_1$$

$$V_o = -R_0 i_L - V_c \implies y = -R_0 x_1 - x_2$$

So the state space representation becomes:

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}-\frac{R_1}{L} & 0\\ \frac{1}{C} & 0\end{bmatrix} x + \begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix} u$$

$$y = \begin{bmatrix}-R_0 & -1\end{bmatrix} x$$

2. The transfer function is given by

$$G(s) = C(sI-A)^{-1}B + D = \begin{bmatrix}-1 & -1\end{bmatrix}\begin{bmatrix}s+1 & 0\\ -1 & s\end{bmatrix}^{-1}\begin{bmatrix}1\\ 0\end{bmatrix} =$$
$$= \frac{1}{(s+1)s}\begin{bmatrix}-1 & -1\end{bmatrix}\begin{bmatrix}s & 0\\ 1 & s+1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = -\frac{s+1}{(s+1)s} = -\frac{1}{s}.$$

There is a pole-zero cancellation at $s=-1$. We can conclude that the system is either not controllable or not observable.

3. In order to compute the zero input transition we need to find the state transition matrix. Therefore we first calculate the eigenvalues of A by solving $\det(\lambda I - A) = 0$. This gives $(\lambda+1)\lambda = 0$, hence the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 0$. Since the eigenvalues are distinct the matrix A is diagonalizable. To diagonalize A we also need the eigenvectors. These satisfy the equations

$$Aw_1 = \lambda_1 w_1 \quad \text{and} \quad Aw_2 = \lambda_2 w_2,$$

which gives $w_1 = [-1 \;\; 1]^T$ and $w_2 = [0 \;\; 1]^T$.

Defining $W := [w_1 \; w_2]$ we obtain the state transition matrix as

$$\Phi(t) = e^{At} = W\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}W^{-1} = \begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}e^{-t} & 0\\ 1-e^{-t} & 1\end{bmatrix}.$$

The zero input transition for $x(0) = [1\ 0]^T$ is then

$$x(t) = e^{At}x(0) = \begin{bmatrix} e^{-t} \\ 1 - e^{-t} \end{bmatrix}.$$

Exercise 4

1a	1b	1c	2a	2b	2c	Exercise
3	2	2	3	7	8	25 Points

1. (a) The system is linear, autonomous and has dimension 1.
   (b) The only equilibrium point of the system is $x = 0$. The system is unstable as $r > 0$.
   (c) The solution of the system is $x(t) = x_0 e^{rt}$. The limes of the solution is $\lim_{t\to\infty} x(t) = \infty$.
2. (a) The system is non-linear, autonomous and has dimension 1.
   (b) Equilibrium points are found by setting the right-hand side of (**??**) to zero:

$$f(x) = rx\left(1 - \frac{x}{K}\right) = 0$$
$$\Rightarrow x = 0 \text{ and } x = K$$

   Linearizing the right hand side of (**??**) yields

$$\tilde{f}(x) = \frac{d}{dx} f(x) = r - \frac{2rx}{K}$$

   and hence, we obtain for the two equilibria

$$x = 0 : \tilde{f}(x) = r \Rightarrow \text{unstable}$$
$$x = K : \tilde{f}(x) = -r \Rightarrow \text{locally asymptotically stable.}$$

   (c) First we show for $t = 0$ that the proposed solution satisfies the initial condition:

$$x(0) = \frac{Kx_0}{K} = x_0$$

   Second, we check if the solution $x(t)$ and it's time-derivative satisfy the differential equation (**??**):

$$\begin{aligned}
\dot{x}(t) &= \frac{d}{dt}\left[\frac{Kx_0 e^{rt}}{K + x_0\left(e^{rt} - 1\right)}\right] \\
&= \frac{Kx_0 r e^{rt}\left(K + x_0\left(e^{rt} - 1\right)\right) - r x_0 e^{rt} K x_0 e^{rt}}{\left(K + x_0\left(e^{rt} - 1\right)\right)^2} \\
&= \frac{Kx_0 r e^{rt}}{\left(K + x_0\left(e^{rt} - 1\right)\right)^2}\left(K - x_0\right)
\end{aligned}$$

   Substitution of $\dot{x}(t)$ and $x(t)$ in (**??**) yields

$$\dot{x}(t) = r x(t)\left(1 - \frac{x(t)}{K}\right)$$
$$\begin{aligned}
\frac{Kx_0 r e^{rt}}{\left(K + x_0\left(e^{rt} - 1\right)\right)^2}\left(K - x_0\right) &= \frac{rKx_0 e^{rt}}{K + x_0\left(e^{rt} - 1\right)}\left(1 - \frac{Kx_0 e^{rt}}{K\left(K + x_0\left(e^{rt} - 1\right)\right)}\right) \\
&= \frac{Kx_0 r e^{rt}}{\left(K + x_0\left(e^{rt} - 1\right)\right)^2}\left(K - x_0\right)
\end{aligned}$$

and hence, the given solution satisfies system (**??**). For $t \to \infty$ we obtain

$$\lim_{t\to\infty} x(t) = K,$$

which is expected since $x = K$ is the only stable equilibrium. For $K \to \infty$, the solution is given by

$$\lim_{K\to\infty} x(t) = x_0 e^{rt},$$

which equals the solution of the unconstrained Malthusian system from (**??**).