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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	Exercise
5	5	10	5	25 Points

1. The system is not autonomous, linear, and consists of 2 states, 2 inputs, and 1 output. The system is not stable since the eigenvalues of A are -1 and 2.

2. The transfer function $G_2(s)$ from the input u_2 to the output y is

$$G_2(s) = \frac{2s-1}{s^2-s-2}.$$

There is a zero at $s = \frac{1}{2}$ and poles at $s = -1,\ 2$.

3. It is indeed possible to reconstruct the time domain description in controllable canonical form since the system is strictly proper. The resulting time domain representation is

$$\dot{\bar{x}}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1(t) \tag{1}$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}(t) \tag{2}$$

The resulting time domain system is not unique, e.g. any number of coordinate changes can be applied. The system is not stable since the eigenvalues of the new state matrix are 1 and -1. The resulting system is controllable, so it is possible to design a stabilizing feedback controller. Lastly, if stabilizing the system and/or tracking a reference output are objectives of the controller, identifying the gain matrix K_2 is not necessary. However, knowledge of K_2 would be important for tracking a reference trajectory of all states of the original system.

4. Reconstructing the time domain description of the system in controllable canonical form requires an additional step since the transfer function is not strictly proper. Given that the transfer function is proper, we can separate the transfer function into a strictly proper transfer function plus a constant. That is, we have that

$$\frac{s+4}{s+2} = \frac{2}{s+2} + 1.$$

Hence, the system in controllable canonical form is

$$\dot{\bar{x}}(t) = -2\bar{x}(t) + u_1(t) \tag{3}$$
$$y(t) = 2\bar{x}(t) + u_1(t). \tag{4}$$

Note that the resulting system in controllable canonical form has one state while the initial system has two states. This is due to the fact that a pole-zero cancellation must be present in the transfer function. While this implies that the SISO system is either not controllable or not observable, it does not imply that the transfer function is incorrect (this is because the application of the feedback control law with gain K_2 may result in an unobservable system). However, the fact that the SISO transfer function is proper does argue against this being the correct transfer function. We see that the measurement $y(t)$ is a function of $u_1(t)$ in the controllable canonical form (i.e., $D = 1$) while in the original problem the output $y(t)$ is not a function of the control input $u_1(t)$ (i.e., $D = 0$). Hence, given that you know the original model of the system and you know the form of the state feedback controller for $u_2(t) = K_2x(t)$, it is most likely that your friend from EPFL is wrong.

Exercise 2

1	2	3	4	Exercise
5	7	7	6	25 Points

1. Consider $u(t) = 0$. The equilibrium points are then determined by $\dot{x}(t) = 0$, or in other words

$$f(x) = \begin{bmatrix} x_2(t) \\ -x_2(t) + \sin x_1(t) \\ -x_3(t) + ax_1(t) \end{bmatrix} = 0. \tag{5}$$

By inspection of (5), and since $0 \leq x_1(t) < 2\pi$ we have that

$$Q_1 = (0,\ 0,\ 0),$$
$$Q_2 = (\pi,\ 0,\ \alpha\pi),$$

are the equilibrium points of the system.

2. To comment on the stability of Q_1, Q_2, we linearize the system around each equilibrium point, and compute the eigenvalues of each Jacobian matrix. Hence, we have

$$A_1 = \frac{\partial f(x)}{\partial x}\Big|_{(0,0,0)} = \begin{bmatrix} 0 & 1 & 0 \\ \cos x_1 & -1 & 0 \\ \alpha & 0 & -1 \end{bmatrix}\Bigg|_{(0,0,0)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ \alpha & 0 & -1 \end{bmatrix},$$
$$A_2 = \frac{\partial f(x)}{\partial x}\Big|_{(\pi,0,\alpha\pi)} = \begin{bmatrix} 0 & 1 & 0 \\ \cos x_1 & -1 & 0 \\ \alpha & 0 & -1 \end{bmatrix}\Bigg|_{(\pi,0,\alpha\pi)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ \alpha & 0 & -1 \end{bmatrix}.$$

The eigenvalues of A_1 are $\lambda_1 = -1$, $\lambda_2 = \frac{-1+\sqrt{5}}{2}$, and $\lambda_3 = \frac{-1-\sqrt{5}}{2}$. We have that $\lambda_2 > 0$, and so Q_1 is an unstable equilibrium point.

Similarly, the eigenvalues of A_2 are $\lambda_1 = -1$, $\lambda_2 = \frac{-1+\sqrt{3}j}{2}$, and $\lambda_3 = \frac{-1-\sqrt{3}j}{2}$. We have that $Re(\lambda_{1,2,3}) < 0$, and so Q_2 is a stable equilibrium point.

3. The linearized system around the origin (i.e. Q_1) is

$$\Delta\dot{x}(t) = A_1\Delta x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \Delta u(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ \alpha & 0 & -1 \end{bmatrix} \Delta x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \Delta u(t),$$
$$\Delta y(t) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \Delta x(t).$$

The controllability matrix is then

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & \alpha & \alpha & -\alpha \end{bmatrix}.$$

The system is controllable if P is full rank ($rank(P) = 3$), i.e. $\alpha \neq 0$.

The observability matrix is

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ \alpha+1 & -1 & -1 \\ -\alpha-1 & \alpha+2 & 1 \end{bmatrix}.$$

By inspection of Q, the system is observable if $\alpha \neq -1$.

4. We choose $g(x(t)) = -\sin x_1(t)$ so as to cancel the nonlinear term which appears in $\dot{x}_2(t)$. The resulting linear system if we apply $u_1(t), u_2(t)$ is

$$\dot{x}(t) = \begin{bmatrix} -k_1 & 1-k_2 & 0 \\ -k_1 & -1-k_2 & 0 \\ \alpha & 0 & -1 \end{bmatrix} x(t).$$

To design gains k_1, k_2 so as to place all eigenvalues of the resulting system to -1, we equate the coefficients of the characteristic polynomial $(\lambda+1)\Big(\lambda^2 + (k_1+k_2+1)\lambda + 2k_1\Big) = 0$, with those of $(\lambda+1)^3 = (\lambda+1)(\lambda^2+2\lambda+1) = 0$. Hence, we have

$$\begin{aligned} k_1 + k_2 + 1 &= 2, \\ 2k_1 &= 1, \end{aligned}$$

which leads to $k_1 = k_2 = \frac{1}{2}$.

Exercise 3

1	2	3	Exercise
7	10	8	25 Points

1. According to the backward Euler method

$$x_{k+1} = x_k + \delta A x_{k+1}$$

we can write:

$$(I - \delta A)\, x_{k+1} = x_k.$$

Since by assumption the matrix $(I - \delta A)$ is invertible, it follows:

$$\begin{aligned} x_{k+1} &= (I - \delta A)^{-1}\, x_k \\ &= \tilde{A} x_k \end{aligned}$$

The discrete system is linear. It is also time invariant and autonomous.

2. Since the matrix A is diagonalizable it can be written in the following form:

$$A = W \Lambda W^{-1}$$

where Λ is a diagonal matrix of eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix A and W is a matrix of corresponding eigenvectors. Then:

$$\begin{aligned} \tilde{A} &= (I - \delta A)^{-1} \\ &= \left(I - W \delta \Lambda W^{-1}\right)^{-1} \\ &= W\,(I - \delta \Lambda)^{-1}\, W^{-1} \\ &= W \begin{bmatrix} \frac{1}{1-\delta\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{1-\delta\lambda_n} \end{bmatrix} W^{-1}. \end{aligned}$$

Notice that the eigenvectors of $\tilde{A}$ are the same as those of the matrix A. The eigenvalues of the matrix $\tilde{A}$ are

$$\tilde{\lambda}_i = \frac{1}{1 - \delta\lambda_i}, \qquad \forall i = 1, \ldots, n.$$

According to Fact 2.17 in the lecture notes, the matrix $(I - \delta A)$ is invertible if and only if its eigenvalues are non-zero, i.e.

$$\begin{aligned} & 1 - \delta\lambda_i \neq 0,\ \forall i \\ \Leftrightarrow \quad & \delta \notin \left\{ \frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n} \right\}. \end{aligned}$$

3. The discrete time system in part (1) is asymptotically stable if and only if

$$\begin{aligned} \left|\frac{1}{1-\delta\lambda_i}\right| &< 1, \qquad \forall i = 1, \ldots, n \\ \Leftrightarrow -1 < \frac{1}{1-\delta\lambda_i} &< 1. \end{aligned}$$

The solution of these inequalities can be written as:

$$\Leftrightarrow \begin{cases} \delta > 0 & , \, if \; \lambda_i < 0 \\ \delta \in \{\emptyset\} & , \, if \; 0 < \lambda_i < \frac{1}{\delta} \\ \delta > \frac{2}{\lambda_i} & , \, if \; 0 < \frac{1}{\delta} < \lambda_i \end{cases}, \quad \forall i.$$

Notice that in the second case, $0 < \lambda_i < \frac{1}{\delta}$, there exist no δ such that the discrete system is asymptotically stable. If there exist $\lambda_i > 0$, then one should choose δ as in the case three, such that $\delta > \frac{2}{\lambda_i}$. Overall, the conditions for δ, in order for the discrete time system in part (1) to be asymptotically stable are:

$$\begin{aligned} &\Leftrightarrow \begin{cases} \delta > 0 & , \, if \; \lambda_i < 0 \\ \delta > \frac{2}{\lambda_i} & , \, if \; \lambda_i > 0 \end{cases} \\ &\Leftrightarrow \delta > \max_{i=1,\cdots,n} \frac{2}{\lambda_i}. \end{aligned}$$

Exercise 4

1	2	3	4	Exercise
7	3	8	7	25 Points

1. Notice that the following equation holds:

$$|d| = \sqrt{x_1^2 + x_3^2}, \qquad |v| = \sqrt{x_2^2 + x_4^2}$$

hence,

$$\cos\varphi = \frac{x_1}{|d|}, \qquad \cos\theta = \frac{x_2}{|v|},$$
$$\sin\varphi = \frac{x_3}{|d|} \qquad \sin\theta = \frac{x_4}{|v|}.$$

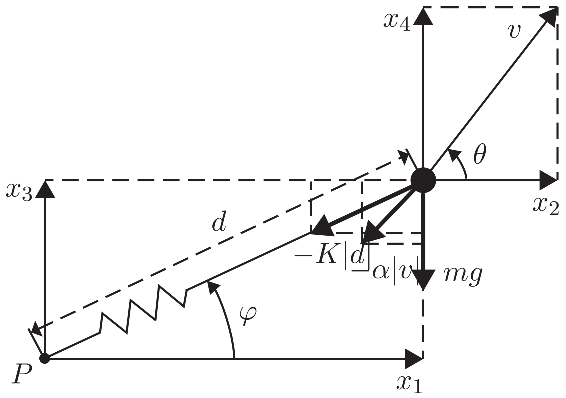

Figure 1: Mass on a spring

The $x_1(x_2)$ and $x_3(x_4)$ components of the spring force and aerodynamic drag force are shown in Figure 1. Along the x_1 (x_2) axis, the relation between x_1 and x_2 can be found using Newton's second law, as follows:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 m = -K|d|\cos\varphi - \alpha|v|\cos\theta.$$

Along the x_3 (x_4) axis, the relation between x_3 and x_4 can be found using Newton's second law, as follows:

$$\dot{x}_3 = x_4$$
$$\dot{x}_4 m = -K|d|\sin\varphi - \alpha|v|\sin\theta - mg.$$

The state space form of the given system is:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{K}{m}x_1 - \frac{\alpha}{m}x_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{K}{m}x_3 - \frac{\alpha}{m}x_4 - g \\
y_1 &= x_1 \\
y_2 &= x_3
\end{aligned}$$

or in the matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -\frac{K}{m} & -\frac{\alpha}{m} & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -\frac{K}{m} & -\frac{\alpha}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} - \begin{bmatrix}0\\ 0\\ 0\\ g\end{bmatrix}$$

$$y = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}x.$$

2. The system is in fact non-linear because of a constant vector $[0\ 0\ 0\ g]^T$. (It is an affine system.) It is time invariant and indeed autonomous.

3. In order to find equilibria of the given system we will set the right hand side to be equal to zero:

$$\begin{cases} x_2 = 0 \\ -\frac{K}{m}x_1 - \frac{\alpha}{m}x_2 = 0 \\ x_4 = 0 \\ -\frac{K}{m}x_3 - \frac{\alpha}{m}x_4 - g = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 = 0 \\ x_1 = 0 \\ x_4 = 0 \\ x_3 = -\frac{mg}{K} \end{cases}$$

Hence, the system has 1 equilibrium which happens when the mass is hanging straight down and spring force exactly balances gravity.
Linearization around the equilibrium point for this system is trivial. The matrix will be the same as one in the original (affine) system:

$$A = \begin{bmatrix}0 & 1 & 0 & 0\\ -\frac{K}{m} & -\frac{\alpha}{m} & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -\frac{K}{m} & -\frac{\alpha}{m}\end{bmatrix}$$

Other way to see this is to introduce the coordinate change $z = x - x_e$, where x_e stands for the equilibrium point. The new system becomes linear with the same system matrix as above.

Then:

$$\Rightarrow \textsc{Det}(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ \frac{K}{m} & \lambda + \frac{\alpha}{m} & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & \frac{K}{m} & \lambda + \frac{\alpha}{m} \end{vmatrix}$$

$$= \begin{vmatrix} \lambda & -1 \\ \frac{K}{m} & \lambda + \frac{\alpha}{m} \end{vmatrix}^2$$

$$= \left(\lambda\left(\lambda + \frac{\alpha}{m}\right) + \frac{K}{m}\right)^2$$

$$= \left(\lambda^2 + \frac{\alpha}{m}\lambda + \frac{K}{m}\right)^2$$

i.e. the system has 2 pairs of eigenvalues on roots of $\lambda^2 + \frac{\alpha}{m}\lambda + \frac{K}{m}$. Since all coefficients in this characteristic polynomial have the same sign, all eigenvalues have a negative real part. Hence, equilibrium is asymptotically stable.

4. The states for which the energy of the system is decreasing can be found by calculating a derivative of the given energy function:

$$\begin{aligned}
\frac{d}{dt}V\left(x(t)\right) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\left(x(t)\right)\frac{d}{dt}x_i(t) \\
&= \frac{\partial V}{\partial x_1}\left(x(t)\right) f_1\left(x(t)\right) + \frac{\partial V}{\partial x_2}\left(x(t)\right) f_2\left(x(t)\right) + \frac{\partial V}{\partial x_3}\left(x(t)\right) f_3\left(x(t)\right) + \frac{\partial V}{\partial x_4}\left(x(t)\right) f_4\left(x(t)\right) \\
&= Kx_1 \cdot x_2 + mx_2\left(-\frac{K}{m}x_1 - \frac{\alpha}{m}x_2\right) + \left(Kx_3 + mg\right)x_4 + mx_4\left(-\frac{K}{m}x_3 - \frac{\alpha}{m}x_4 - g\right) \\
&= -\alpha\left(x_2^2 + x_4^2\right)
\end{aligned}$$

Hence, the energy of the system is decreasing for all states for which the velocity of the mass is different from zero, i.e. $x_2^2 + x_4^2 \neq 0$.