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D-ITET
Spring 2010
28.08.2010

Signal and System Theory II
4. Semester, BSc

Solutions

1 Exercise 1

1	2	3	4	Exercise
5	7	7	6	25 Points

1. The controllability matrix of the system is given by $P = [B\ AB]$, so
$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
Since $rank(P) = 2$, the system is controllable.
The observability matrix of the system is given by $Q = \begin{bmatrix} C \\ CA \end{bmatrix}$, so
$$Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
The system is unobservable since $rank(Q) = 1$.

2. The system is controllable, so the reachable set is the whole $\mathbb{R}^2$. Since the system is unobservable, the unobservable subspace is given by the null space $N(Q)$ of the observability matrix. It can be easily shown that $N(Q) = span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$.

3. Consider the feedback $u(t) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Then the closed loop system is given by
$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$
The characteristic polynomial of the closed loop system is
$$\lambda^2 + k_2\lambda + k_1 = 0.$$
Having both poles at -1 implies that this should be the same as $(\lambda + 1)^2 = \lambda^2 + 2\lambda + 1 = 0$. Equating coefficients leads to $k_1 = 1$ and $k_2 = 2$.

4. We have that $\dot{x}_2(t) = u(t)$. Hence, for $0 \le t < 1$
$$x_2(t) = a_1 t + x_2(0).$$
From the first equation of the system $\dot{x}_1(t) = x_2(t)$ we get that
$$x_1(t) = a_1 \frac{t^2}{2} + x_2(0)t + x_1(0).$$
Since $x_1(0) = 1$, and $x_2(0) = 0$, we have that
$$x_1(t) = a_1 \frac{t^2}{2} + 1,$$
$$x_2(t) = a_1 t.$$

For $t \geq 1$ we have

$$\begin{aligned} x_2(t) &= a_2(t-1) + x_2(1), \\ x_1(t) &= a_2\frac{(t-1)^2}{2} + x_2(1)(t-1) + x_1(1). \end{aligned}$$

By continuity of x_2, $x_2(1) = a_1$. Similarly for x_1, $x_1(1) = \frac{a_1}{2} + 1$. By summarizing the results for $t \geq 1$ we get

$$\begin{aligned} x_2(t) &= a_2(t-1) + a_1, \\ x_1(t) &= a_2\frac{(t-1)^2}{2} + a_1(t-1) + \frac{a_1}{2} + 1. \end{aligned}$$

Since $x_1(2) = 0$, and $x_2(2) = 2$, for $t = 2$ we get

$$\begin{aligned} &a_2 + a_1 = 2, \\ &\frac{1}{2}a_2 + \frac{3}{2}a_1 + 1 = 0. \end{aligned}$$

From the last set of equations we can compute $a_1 = -2$, and $a_2 = 4$.

2 Exercise 2

1	2	3	4	Exercise
5	6	6	8	25 Points

1. The system is linear and time invariant. The eigenvalues of the state matrix A are $\lambda_1 = -3$ and $\lambda_2 = 2$. Therefore, since one of the eigenvalues is strictly greater than zero, the system is not stable.

2. The transfer function from the controlled input to the output is:
$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B_u = \frac{3}{s-2}.$$
The transfer function from the uncontrolled disturbance input to the output is:
$$H(s) = \frac{Y(s)}{W(s)} = C(sI - A)^{-1}B_w = \frac{1}{s+3}.$$

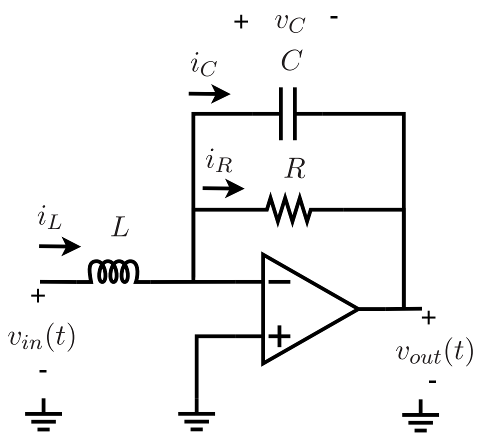
3. The nominal system is uncontrollable, so normaly one would not expect stabilization by state feedback to be possible. However, note that the unstable eigenvalue of A appears in the transfer function from $U(s)$ to $Y(s)$. Hence the unstable mode is controllable and the system is stabilizable.

4. Applying the state feedback controller to the nominal system we obtain the closed loop system:
$$\dot{x}(t) = \begin{bmatrix} -1 & -5 \\ 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$
The resulting closed loop state matrix $A + BK$ has eigenvalues at $\lambda_1 = -3$ and $\lambda_2 = -1$ and is therefore stable. The transfer function from the uncontrolled disturbance input to the output is now:
$$H(s) = \frac{Y(s)}{W(s)} = C(sI - A - B_uK)^{-1}B_w = \frac{s-8}{s^2+4s+3}.$$
Since the Laplace Transform of a step is equal to $\frac{1}{s}$
$$Y(s) = \frac{s-8}{s^2+4s+3}W(s) = \frac{s-8}{s^2+4s+3} \cdot \frac{1}{s}.$$
Applying the Final Value Theorem, we have that
$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s) \\ &= \lim_{s\to 0} \frac{s-8}{s^2+4s+3} \\ &= \frac{-8}{3}. \end{aligned}$$

3 Exercise 3

1	2	3	Exercise
8	10	7	25 Points

1. Let us consider v_C, the voltage on the capacitor, and i_L, the current on the inductor, as our states.



$$v_{out} = -v_C$$

$$v_{in} = L\frac{d}{dt}i_L \Rightarrow \frac{d}{dt}i_L = \frac{1}{L}v_{in}$$

we also have

$$v_C = \frac{1}{C}\int i_C dt \Rightarrow \frac{d}{dt}v_C = \frac{1}{C}i_C$$

but

$$i_C = i_L - \frac{1}{R}v_C$$

hence,

$$\frac{d}{dt}v_C = \frac{1}{C}i_L - \frac{1}{RC}v_C$$

and finally,

$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_{in}$$

$$v_{out} = \begin{bmatrix} -1 & 0 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + 0v_{in}$$

For the transfer function we have:

$$\begin{bmatrix} sV_C \\ sI_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_C \\ I_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V_{in}$$

and $V_{out} = -V_C$

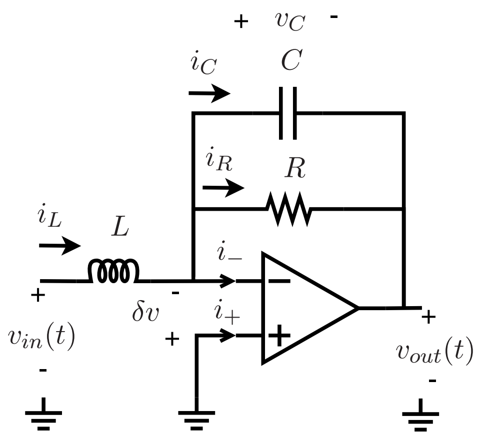
resulting in $sV_C = -\frac{V_C}{RC} + \frac{I_L}{C}$ and $sI_L = \frac{V_{in}}{L}$ which gives

$$\frac{V_{out}}{V_{in}} = -\frac{1}{sLC\left(s + \frac{1}{RC}\right)} = -\frac{1}{LCs^2 + \frac{L}{R}s}$$

2. We can write the transfer function as follows

$$V_{out} = \frac{K}{s+1}\delta V$$

From this transfer function we can go back to the time domain and we get that



$$\frac{d}{dt}v_{out} = K\delta v - v_{out} \tag{1}$$

Augmenting the state with v_{out}, the new state is $x^T = \begin{bmatrix} v_C & i_L & v_{out} \end{bmatrix}$ we also have the following equations:

$$v_{out} = -v_C - \delta v \Rightarrow \delta v = -v_C - v_{out} \tag{2}$$

$$v_{in} = L\frac{d}{dt}i_L - \delta v \Rightarrow \frac{d}{dt}i_L = \frac{1}{L}v_{in} + \frac{1}{L}\delta v \tag{3}$$

now, substituting (2) in (3) we have

$$\frac{d}{dt}i_L = \frac{1}{L}v_{in} - \frac{1}{L}v_C - \frac{1}{L}v_{out}$$

and substituting (2) in (1) we have

$$\frac{d}{dt}v_{out} = -(K+1)\,v_{out} - Kv_C$$

the equation for $\frac{d}{dt}v_C$ remains as before, because the currents are the same (note that $i_- = i_+ = 0$)

$$\frac{d}{dt}v_C = \frac{1}{C}i_L - \frac{1}{RC}v_C$$

the state space representation is then given by:

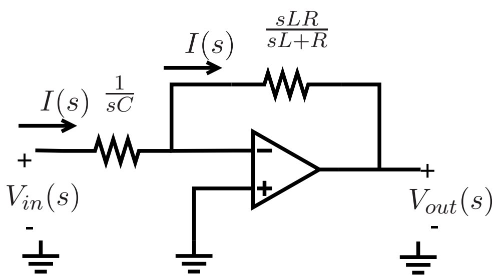
$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \\ v_{out} \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} & 0 \\ -\frac{1}{L} & 0 & -\frac{1}{L} \\ -K & 0 & -(K+1) \end{bmatrix} \begin{bmatrix} v_C \\ i_L \\ v_{out} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \\ 0 \end{bmatrix} v_{in}$$

$$v_{out} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \\ v_{out} \end{bmatrix} + 0 v_{in}$$

3. By making use of the impedance method we can write the circuit in the following form:

$$V_{in}(s) = \frac{1}{sC} I(s)$$

$$V_{out}(s) = -\frac{sLR}{sL+R} I(s)$$



Hence the transfer function is:

$$\frac{V_{out}}{V_{in}} = -\frac{LRCs^2}{sL+R} \tag{4}$$

This transfer function is not proper (degree of the numerator is higher than the degree of the denominator). This is due to the fact that this circuit is a differentiator, it differentiates the input. Hence it cannot be put in the state space form, where only the input, but not its derivative can appear at the output.

4 Exercise 4

1	2	3	4	Exercise
6	7	7	5	Points

1. $x(t)$ can be written as (cf. script slide 6.6, Sampled Data Linear Systems):

$$x(t) = e^{A(t-kT)}x(kT) + \int_{kT}^{t} e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau$$

$$x((k+1)T) = e^{AT}x(kT) + \left(\int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]}B\mathrm{d}\tau \right) u_k$$

The integral on the right hand side can be reformulated using the substitution $s = \tau - kT$ to

$$\int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]}B\mathrm{d}\tau = \int_{0}^{T} e^{A(T-s)}B\mathrm{d}s$$

With this we finally get

$$x((k+1)T) = e^{AT}x(kT) + \left(\int_{0}^{T} e^{A(T-\tau)}B\mathrm{d}\tau \right) u_k \qquad (5)$$

So, $\hat{A} = e^{AT}$ and $\hat{B} = \int_0^T e^{A(T-\tau)}B\mathrm{d}\tau$.

2. As A is diagonalizable we can write $A = W\Lambda W^{-1}$ where W is a matrix of eigenvectors and Λ a diagonal matrix with eigenvalues

$$\Lambda = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

Hence,

$$\begin{aligned} \hat{A} &= e^{AT} \\ &= I + AT + \ldots + \frac{1}{k!}(AT)^n + \ldots \\ &= I + W\Lambda W^{-1}T + \ldots + \frac{1}{k!}\left(W\Lambda W^{-1}\right)^k T^k + \ldots \end{aligned}$$

Note, that $I = WW^{-1}$ and $\left(W\Lambda W^{-1}\right)^k = W\Lambda^k W^{-1}$, therefore

$$\hat{A} = W \underbrace{\left(I + \Lambda T + \ldots + \frac{1}{k!}\Lambda^k T^k + \ldots \right)}_{e^{\Lambda T}} W^{-1}$$

As a result, if A is diagonalizable, then $\hat{A}$ is also diagonalizable with the eigenvalues

$$\hat{\lambda}_i = e^{\lambda_i T} \tag{6}$$

3. If A is diagonalizable, then $\hat{A}$ is also diagonalizable as shown above. Moreover, we have eigenvalues
$$\hat{\lambda}_i = e^{\lambda_i T}$$
The discrete time system is asymptotically stable, if and only if $\left|\hat{\lambda}_i\right| < 1$, for all $i = 1, \ldots, n$, or equivalently $\left|e^{\lambda_i t}\right| < 1$.

   With $\lambda_i = \sigma_i \pm j\omega_i$ this is equivalent to
$$\left|e^{\lambda_i T}\right| = \left|e^{(\sigma_i \pm j\omega_i)T}\right| = \left|e^{\sigma_i T}\right| \left|e^{\pm j\omega_i T}\right| < 1$$
Since $\left|e^{\pm j\omega_i T}\right| = 1$, the inequality is satisfied if and only if σ_i is less than zero. This is equivalent to
$$Re\{\lambda_i\} < 0$$
Hence the continuous time system is asymptotically stable, if and only if the discrete system is asymptotically stable.

4. To have $x_k = 0$ for all $k \geq n$, we aim for a deadbeat respofnse and a *nilpotent matrix* $\hat{A}$ (cf. slide 6.13). Since A is diagonalizable, $\hat{A}$ is also diagonalizable. Moreover $\hat{\lambda}_i = e^{\lambda_i T} \neq 0$ for all finite λ_i. This contradicts the condition for nilpotent matrices, where all eigenvalues have to be zero. Therefore, you should not believe your friend from EPFL.