Automatic Control Laboratory
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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

1 Exercise 1

1	2	3	4	Exercise
5	5	7	8	25 Points

The following differential equation was proposed by Euler to model the buckling of a flexible beam.

$$m\ddot{\theta} + d\dot{\theta} - \mu\theta + \lambda\theta + \theta^3 = 0\,, \tag{1}$$

where m, d, μ, λ are positive constant parameters.

1. What is the dimension of the system? Is the system autonomous? Is it linear?
2. Write the system in state space form using $x_1 = \theta$ and $x_2 = \dot{\theta}$ as states.
3. Determine all the equilibria of the system for the cases $\mu > \lambda$ and $\mu < \lambda$.
4. Using linearization, determine the stability of the equilibrium $\hat{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for the cases $\mu > \lambda$ and $\mu < \lambda$.

2 Exercise 2

1	2	3	4	Exercise
5	8	4	8	25 Points

Consider the discrete time, linear, time invariant system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \qquad x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m, y \in \mathbb{R}^p \\ y_k &= Cx_k + Du_k \qquad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}\,. \end{aligned}$$

1. Assume that $x_0 = 0$ (zero state response). Show by induction that for $k = 1, 2, \ldots$

$$x_k = \sum_{i=0}^{k-1} A^{k-i-1} B u_i\,.$$

2. A state $x \in \mathbb{R}^n$ is called reachable from 0 if there exists a sequence $u_0, u_1, \ldots, u_{n-1}$ such that $x_n = x$ starting from $x_0 = 0$. Show that x is reachable if and only if it is in the range space of the matrix:

$$P = [B \;\; AB \; \ldots \; A^{n-1}B] \in \mathbb{R}^{n\times n\cdot m}\,.$$

3. Assume now that $x_0 \neq 0$ but $u_k = 0$ for all $k = 0, 1, 2, \ldots$ (zero input response). Show that for $k = 1, 2, \ldots$

$$x_k = A^k x_0\,.$$

4. A state $x_0 \in \mathbb{R}^n$ is called unobservable if the zero input response is such that $y_0 = y_1 = \cdots = y_{n-1} = 0$. Show that a state is unobservable if and only if it is in the nullspace of the matrix $Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{p\cdot n\times n}$.

3 Exercise 3

1	2	3	4	Exercise
6	6	7	6	25 Points

Consider the continuous time, linear, time invariant system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x\,. \end{aligned}$$

1. Compute the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Is the matrix diagonalizable?
2. Is the system controllable? Is it observable?
3. Show that the state transition matrix is given by:

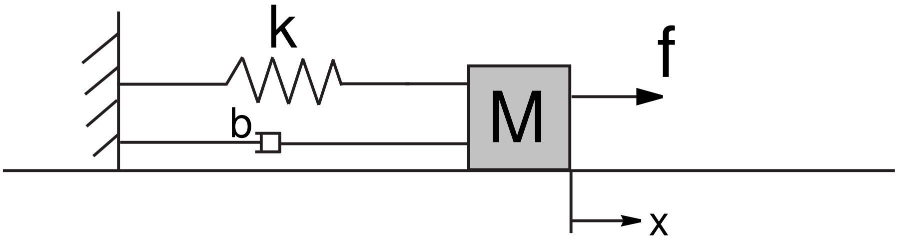
$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Compute the Laplace transform of the state transition matrix. Hence, or otherwise, compute the transfer function of the system.

Exercise 4

1	2	3	4	Exercise
5	5	6	9	25 Points

Consider the mechanical system of the following figure. The mass M is attached to a wall by means of a spring of stiffness k and a damper of constant b. The damper exerts a force proportional to the speed of the mass and in the opposite direction. An external force f is applied to the mass. Let x denote the horizontal displacement of the mass from its equilibrium point and $v = \frac{dx}{dt}$ its velocity.



1. Write the state space model for this system using $x_1 = v$, $x_2 = x$ as states, $u = f$ as input and $y = x$ as output.
2. Using the values $M = 1$Kg, $b = 2$Kg/sec and $k = 5$Nt/m, derive the transfer function $G(s)$ from the input u to the output y.
3. Assume that the force applied to the mass is $f(t) = 10(1 - e^{-t})$Nt, $t \geq 0$. For the parameter values of part 2 of this exercise, calculate the steady state value $y_\infty = lim_{t\to\infty} y(t)$ by using the Final Value Theorem.
4. A step input of magnitude $f = 10$Nt, $t \geq 0$ is now applied to the system. Calculate the expression of $y(t)$ as a function of time. Use the parameter values of part 2 of this exercise and zero initial condition ($x_1(0) = 0$ and $x_2(0) = 0$).