# Signals and Systems II Videos

# Introduction to Modeling

01 Modeling

# Why modelling?

PHYSICAL SYSTEM $\xRightarrow{\text{WHY?}}$ MATHEMATICAL MODEL

TIME $\longrightarrow$ $t \geq 0$

STATE $\longrightarrow$ $x(t) \in \mathbb{R}^n$

OUTPUT $\longrightarrow$ $y(t) \in \mathbb{R}^p$

INPUT $\longrightarrow$ $u(t) \in \mathbb{R}^m$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = \underbrace{Cx(t) + Du(t)}_{\text{LINEAR } (x(t),\, u(t))}$$

$\Downarrow$

LINEAR SYSTEM.

$$n, p, m \in \mathbb{N}$$

$$A \in \mathbb{R}^{n \times n}$$

$$B \in \mathbb{R}^{n \times m}$$

$$C \in \mathbb{R}^{p \times n}$$

$$D \in \mathbb{R}^{p \times m}$$

- Predict future evolution
- Determine properties
- Steer using inputs, …

# Basic steps

1. Identify input variables $u(t) \in \mathbb{R}^m$
   - Quantities that come from outside the system
2. Identify output variables $y(t) \in \mathbb{R}^p$
   - Quantities that can be measured
3. Select state variables $x(t) \in \mathbb{R}^n$
   - Related to "energy storage"
4. Compute derivatives of the states
   - Physical laws, chemical laws, …
   - Write derivatives in terms of states and inputs
5. Write outputs in terms of states and inputs $y(t) = Cx(t) + Du(t)$

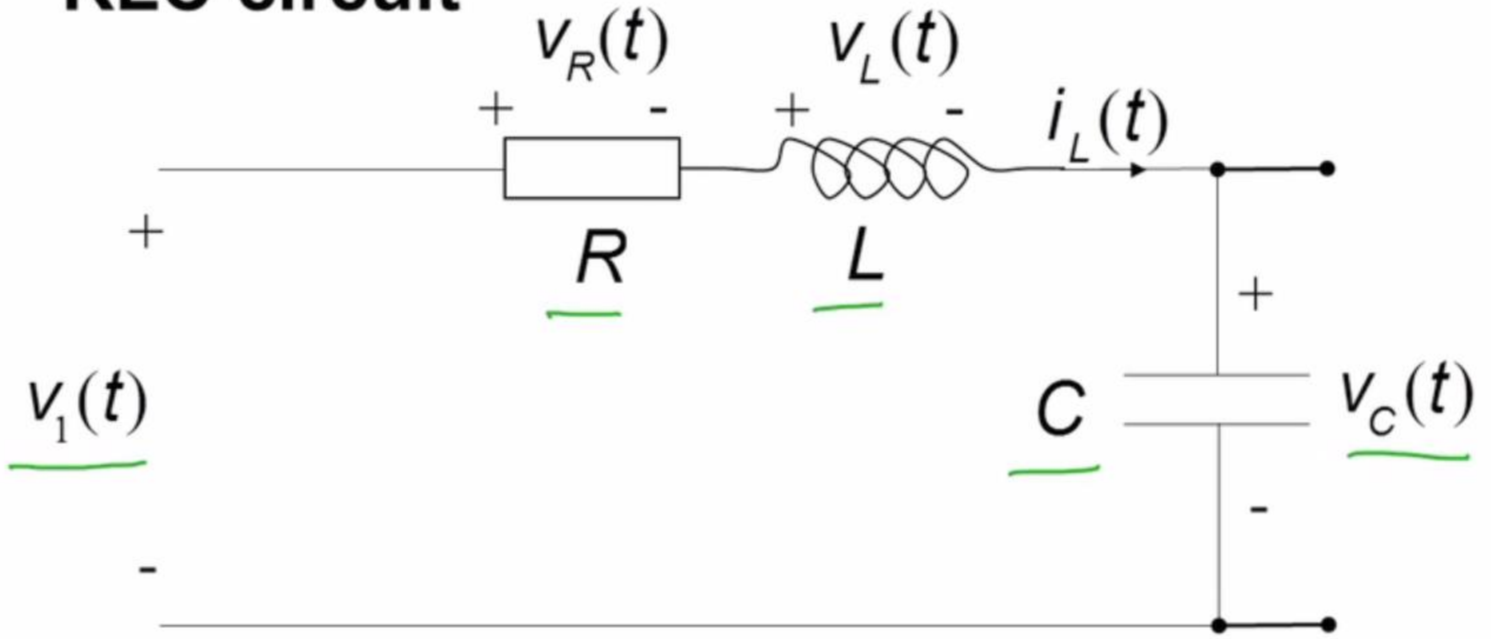
# Disclaimes

- Seems easy, but some skill is necessary
  - State selection ✓
  - Dynamical equations ✓
- Mathematical model NEVER the same as reality
- With any luck, close enough to be useful!

# Modeling an electric circuit

01 Modeling

# RLC circuit

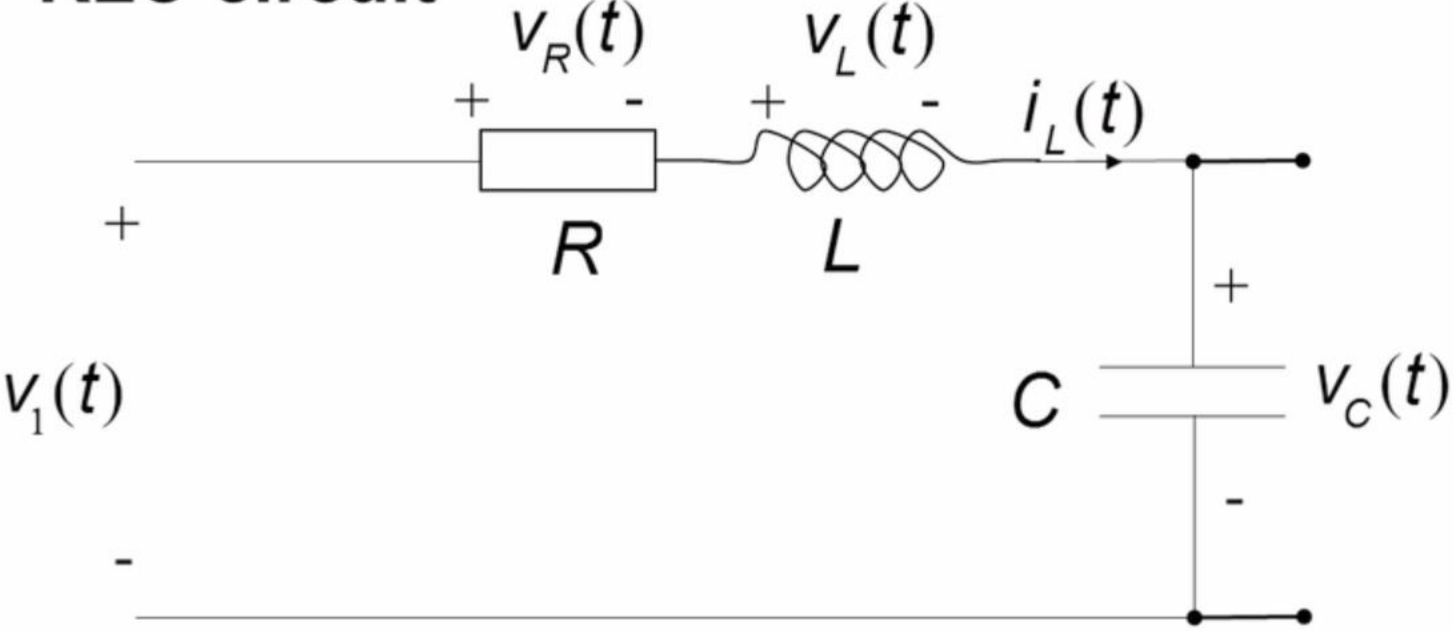



- Given
  - $v_1(t), \; t \geq 0$
  - $v_C(0)$, $i_L(0)$ } ENERGY IN CIRCUIT
- Find
  - $v_C(t)$, $i_L(t)$, $v_L(t)$, $v_R(t)$ } $t \geq 0$
- Model
  - Equations relating these
  - Solve to determine evolution

- Inputs: $u(t) = v_1(t) \in \mathbb{R}$ ← m=1
- Outputs: $y(t) = v_C(t) \in \mathbb{R}$ ← p=1
- States: ? $x_1(t), \ldots, x_n(t) \in \mathbb{R}$

$$\dot{x}_i(t) = a_{i1} x_1(t) + \ldots + a_{in} x_n(t) + b_i u(t)$$

# RLC circuit




- Element equations

$$\left.\begin{aligned} C\frac{dv_C(t)}{dt} &= i_L(t) \\ L\frac{di_L(t)}{dt} &= v_L(t) \end{aligned}\right\} (*)$$

$$v_R(t) = Ri_L(t)$$

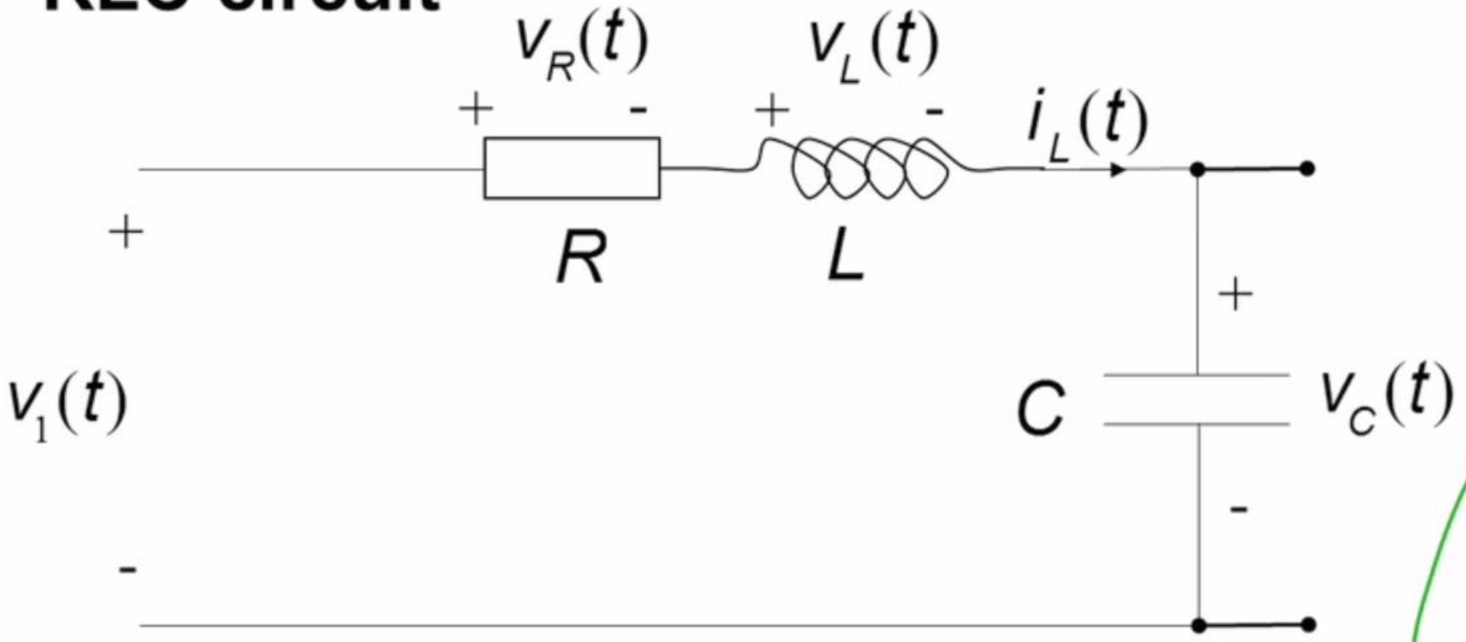
$(*)$ $x_1(t) = v_c(t)$

$x_2(t) = i_L(t)$

$$\dot{x}_1(t) = \frac{1}{C}x_2(t)$$

$$\dot{x}_2(t) = \frac{1}{L}v_L(t)$$

# RLC circuit




- So far

$$\dot{x}_1(t) = \frac{1}{C} x_2(t)$$

$$\dot{x}_2(t) = \frac{1}{L} v_L(t)$$

$$v_R(t) = R \underbrace{x_2(t)}_{i_L(t)}$$

$$\dot{x}_2(t) = \frac{1}{L}\left(u(t) - v_R(t) - x_1(t)\right)$$

$$\boxed{\dot{x}_2(t) = \frac{1}{L} u(t) - \frac{R}{L} x_2(t) - \frac{1}{L} x_1(t)}$$

- Kirchoff's Laws

$$v_L(t) = \cancel{v_1}^{\,u(t)}(t) - v_R(t) - \cancel{v_C}^{\,x_1(t)}(t)$$

# RLC circuit

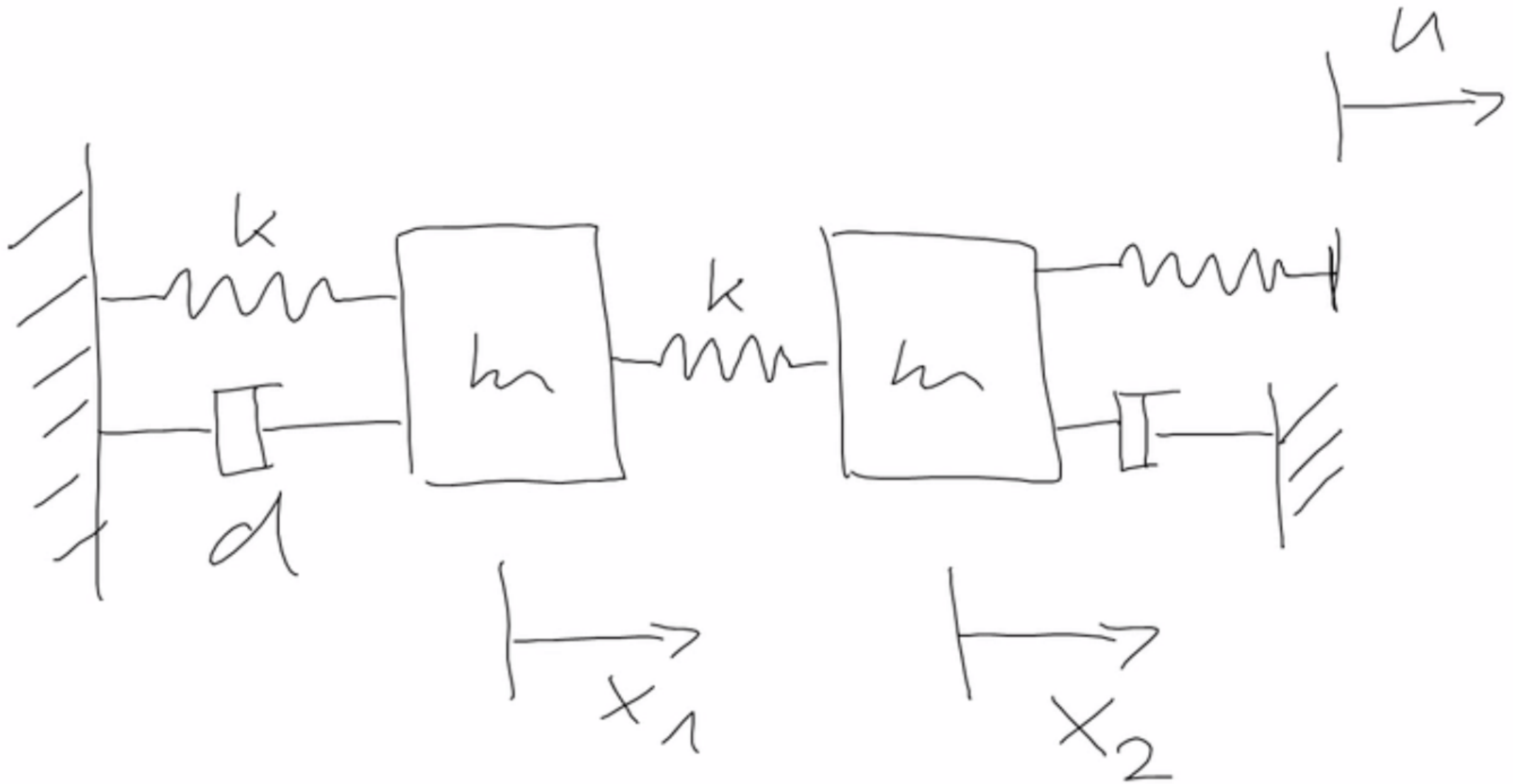
$$\dot{x}(t) = \overbrace{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}}^{A \in \mathbb{R}^{2\times 2}} x(t) + \overbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}^{B \in \mathbb{R}^{2\times 1}} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C \in \mathbb{R}^{1\times 2}} x(t) + \underbrace{\cancel{D}}_{D = 0 \in \mathbb{R}^{1\times 1}} u(t)$$

- Note: States related to energy storage → $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} \in \mathbb{R}^2$ $\quad n = 2$
- Generally a good idea!

# Modeling double-mass dynamics

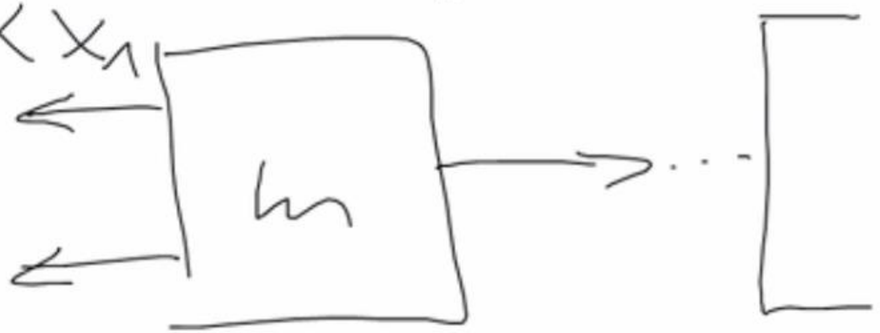
01 Modeling

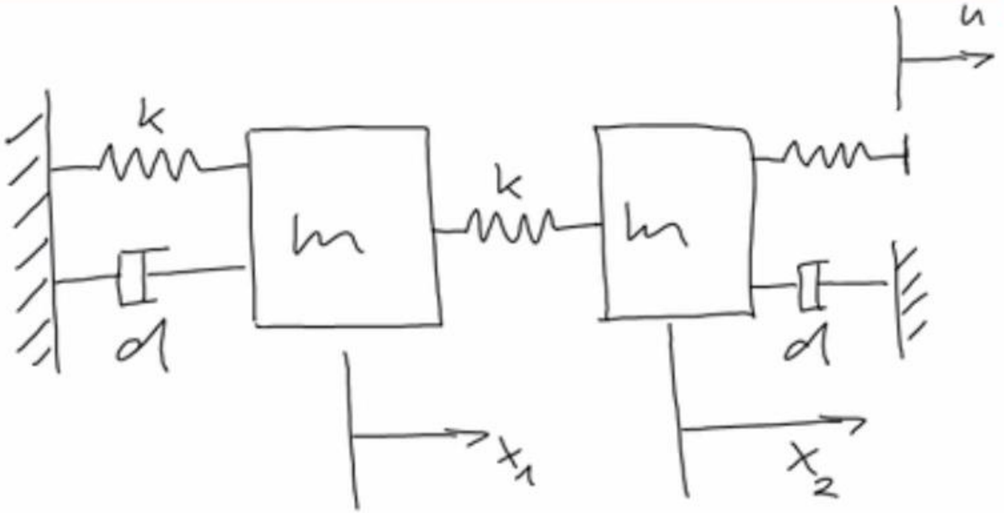

u
k
k
m
m
d
$x_1$
$x_2$

## 1st mass

$$F_{k_2} = k(x_2 - x_1)$$

$$F_{k_1} = k x_1$$

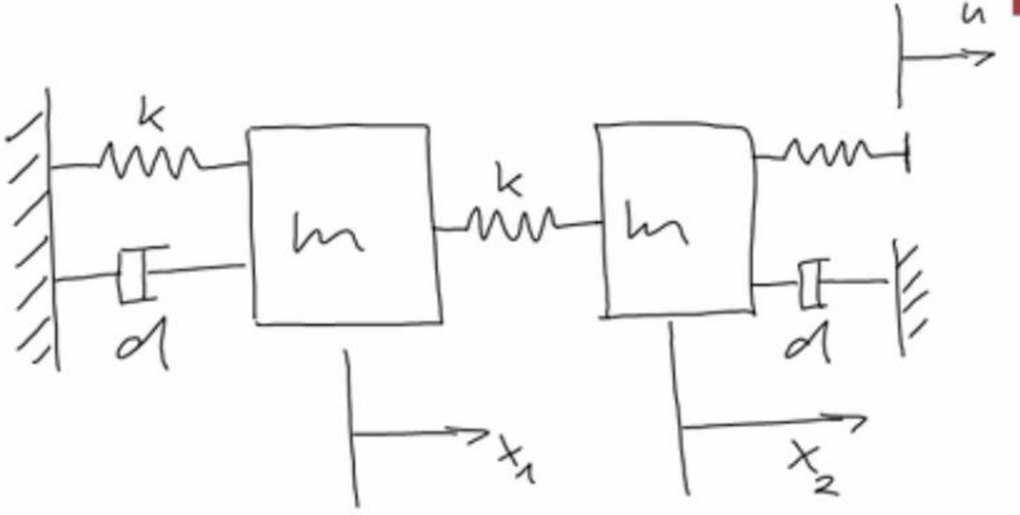



$$F_{d_1} = d\dot{x}_1$$

$$m\ddot{x}_1 = -kx_1 - d\dot{x}_1 + k(x_2 - x_1)$$

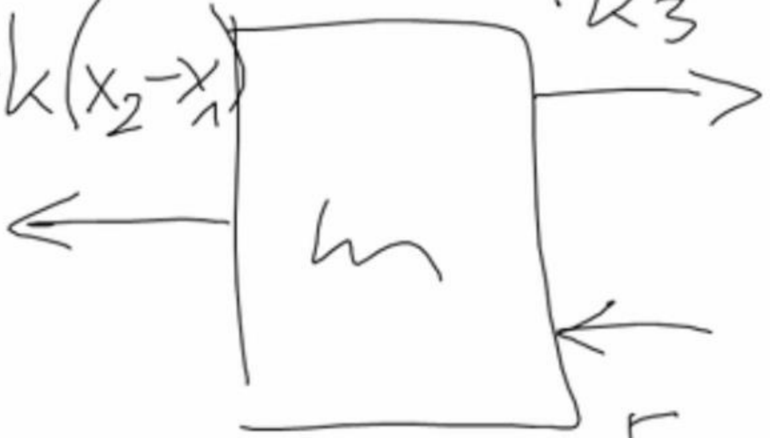
# 2nd mass




$$m\ddot{x}_1 = -kx_1 - d\dot{x}_1 + k(x_2 - x_1)$$

$$F_{k3} = k(u - x_2)$$

$$F_{k2} = k(x_2 - x_1)$$

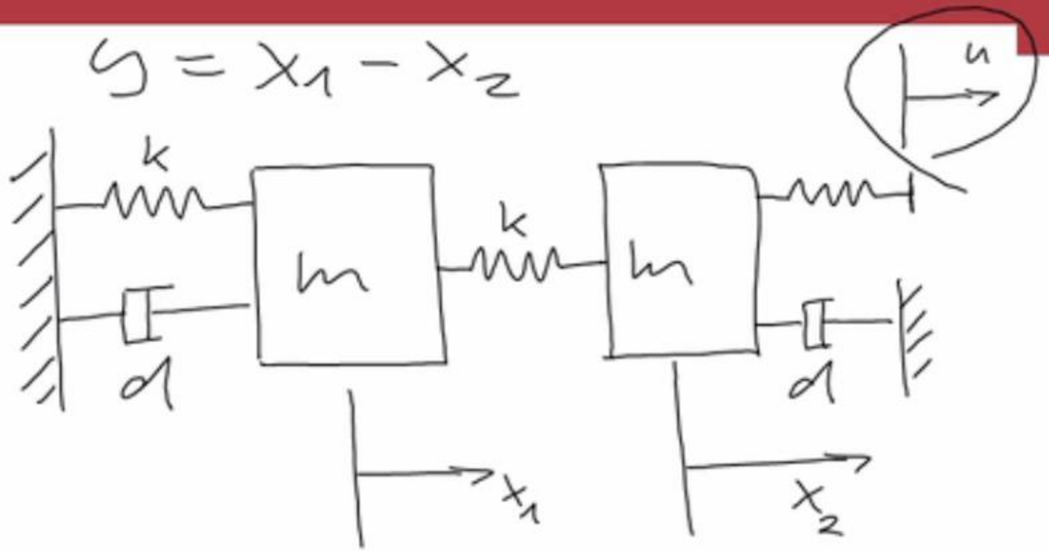



$$F_{d2} = d\dot{x}_2$$

$$m\ddot{x}_2 = -k(x_2 - x_1) - d\dot{x}_2 + k(u - x_2)$$

# Linear state space repr.

$$y = x_1 - x_2$$




$$\dot{x} = Ax + Bu$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2k}{m} & -\frac{d}{m} & \frac{k}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m} & 0 & -\frac{2k}{m} & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{m} \end{bmatrix} [u]$$

$$m\ddot{x}_1 = -kx_1 - d\dot{x}_1 + k(x_2 - x_1)$$

$$m\ddot{x}_2 = -k(x_2 - x_1) + k(u - x_2) - d\dot{x}_2$$

$$y = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$$

# LinAlg Revision: Linear Equations

02 ODEs and Linear Algebra

## Systems of linear equations

$$Ax = y \qquad A \in \mathbb{R}^{n\times m}, y \in \mathbb{R}^n \text{ given}$$
$$x \in \mathbb{R}^m \text{ unknown}$$

- Many interpretations: e.g., y are sensor measurements/outputs, x are inputs or model parameters, A is a linear model relating inputs to outputs
- m=n: unique solution iff A is invertible $A^{-1}Ax = A^{-1}y \Rightarrow x = A^{-1}y$
- n>m: more equations than unknowns (overdetermined), no solution in general
  - Find x that minimizes ||Ax – y||: if A is rank m, then

$$x = \left(A^T A\right)^{-1} A^T y$$

- n<m: fewer equations than unknowns (underdetermined), infinitely many solutions
  - Find x with minimum norm: if A is rank n, then

$$x = A^T \left(AA^T\right)^{-1} y$$

# LinAlg Revision: The 2 Norm

02 ODEs and Linear Algebra

The 2-norm is a measure of "size" or "length"

**Definition**: The 2-norm is a function $\|\bullet\|:\mathbb{R}^n \to \mathbb{R}$ that to each $x \in \mathbb{R}^n$ assigns a real number

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \|x\| = \sqrt{x_1^2 + x_2^2}$$

$$\|x\|^2 = \left(\sqrt{x_1^2 + x_2^2}\right)^2 = x_1^2 + x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x^T x$$

## Some important facts about the 2-norm

**Fact 2.1**: For all $x, y \in \mathbb{R}^n$, $a \in \mathbb{R}$

1. $\|x\| \geq 0$ and $\|x\| = 0$ if & only if $x = 0$
2. $\|ax\| = |a| \cdot \|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

$$\|-2x\| = \sqrt{(-2x_1)^2 + (-2x_2)^2}$$
$$= \sqrt{4x_1^2 + 4x_2^2}$$
$$= \sqrt{4(x_1^2 + x_2^2)} = 2\|x\|$$

$|-2|$

Distance between $x, y \in \mathbb{R}^n$ is $\|x - y\|$

$$x = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}$$

$$\|x - y\| = \left\| \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} \right\| = \sqrt{25 + 9 + 0}$$

$$= \sqrt{34}$$

# LinAlg Revision: Linear Independance

02 ODEs and Linear Algebra

**Definition**: A set of vectors $\{x_1, x_2, \cdots x_m\} \in \mathbb{R}^n$ is called linearly independent if for $a_1, a_2, \cdots a_m \in \mathbb{R}$

$$a_1 x_1 + a_2 x_2 + \cdots + a_m x_m = 0 \Leftrightarrow a_1 = a_2 = \cdots = a_m = 0$$

Otherwise they are called linearly dependent.

$$x_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$a_1 x_1 + a_2 x_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a_1 \cdot (2) + a_2 \cdot (1) = 0 \quad (*)$$

$$\boxed{a_2 \cdot (3) = 0 \quad (**)} \rightarrow \underline{a_2 = 0}$$

$$(*) \rightarrow a_1 \cdot (2) = 0 \rightarrow \underline{a_1 = 0} \longrightarrow$$ linearly independent

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**Definition**: A set of vectors $\{x_1, x_2, \cdots x_m\} \in \mathbb{R}^n$ is called linearly independent if for $a_1, a_2, \cdots a_m \in \mathbb{R}$

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_m x_m = 0 \Leftrightarrow a_1 = a_2 = \cdots = a_m = 0}$$

Otherwise they are called linearly dependent.

$$x_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$a_1 x_1 + a_2 x_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a_1 \cdot 2 + a_2 \cdot 1 = 0 \quad (*) \longrightarrow a_2 = -2a_1$$

$$a_1 \cdot 4 + a_2 \cdot 2 = 0 \quad (**) \longrightarrow a_2 = -2a_1$$

$$a_1 = 1, \; a_2 = -2$$

$$x_1 = 2x_2 \rightarrow$$ linearly dependent

**Fact 2.3**: There exists a set with $n$ linearly independent vectors in $\mathbb{R}^n$, but any set with more than $n$ vectors is linearly dependent.

**Exercise**: What is a set of $n$ linearly independent vectors of $\mathbb{R}^n$?

$n = 3$

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\rightarrow a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$

$\rightarrow$ $a_1 = 0$, $a_2 = 0$, $a_3 = 0$ $\Rightarrow$ linearly independent

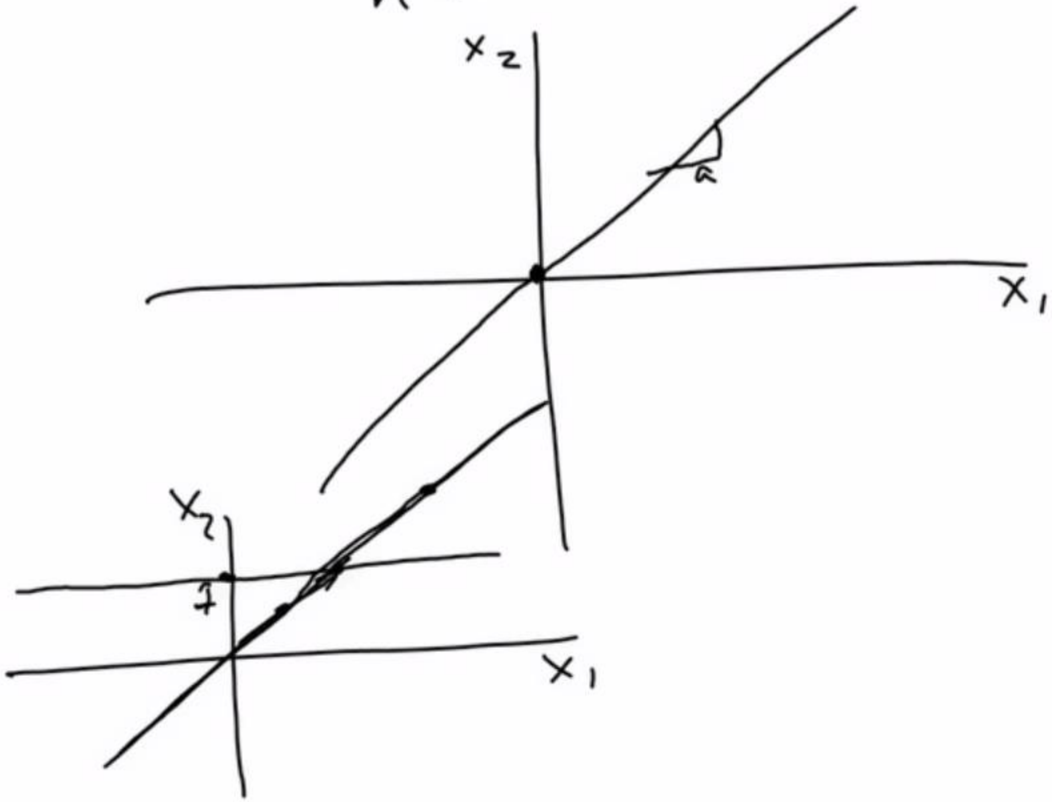
# LinAlg Revision: Subspaces

02 ODEs and Linear Algebra

## Subspaces

> **Definition**: A set of vectors $S \subseteq \mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if for all $x, y \in S$ and $a, b \in \mathbb{R}$, we have that $ax + by \in S$.

$n = 2$




- Generally an infinite set
- Some examples:
  - $S = \mathbb{R}^n$, $S = \{0\}$
  - $\{x \in \mathbb{R}^n \mid x_2 = a x_1\}$
- Some example that are not subspaces

  $\{x \in \mathbb{R}^n \mid x_2 = 1\}$

## Basis of a Subspace

**Definition**: The span of $\{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}^n$ is set of all linear combinations of these vectors

**Definition**: A set of vectors $\{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}^n$ is called a basis for a subspace $S \subseteq \mathbb{R}^n$ if

1. $\{x_1, x_2, \ldots, x_m\}$ are linearly independent
2. $S = \text{span}\{x_1, x_2, \ldots, x_m\}$

In this case, $m$ is called the dimension of $S$.

- All subspaces of $\mathbb{R}^n$ have bases, though not unique
- Different bases related through coordinate transformation

# LinAlg Revision: Range and Null Space

02 ODEs and Linear Algebra

## Range space of a matrix

**Definition**: The range space of a matrix $A \in \mathbb{R}^{n\times m}$ is the set

$$\text{range}(A) = \{y \in \mathbb{R}^n \mid \exists\, x \in \mathbb{R}^m, y = Ax\}$$

- Fact: range(A) is a subspace of $\mathbb{R}^n$

- **Definition**: The rank of a matrix $A \in \mathbb{R}^{n\times m}$ is the dimension of range$(A)$.

columns of A

- Fact: range(A) = span$\{a_1,\ldots,a_m\}$,
  so rank(A) is the number of linearly independent columns of A

Example:

Is $\underbrace{\begin{bmatrix}1\\3\\4\end{bmatrix}}_{y}$ in the range of $A = \begin{bmatrix}1 & 0\\2 & 1\\3 & 1\end{bmatrix}$?

$$A \underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{x} = \underset{y}{\begin{bmatrix}1\\3\\4\end{bmatrix}}$$

yes

## Null space of a matrix

**Definition**: The null space of a matrix $A \in \mathbb{R}^{n \times m}$ is the set

$$\text{null}(A) = \{ x \in \mathbb{R}^m \mid Ax = 0 \}$$

- Fact: null(A) is a subspace of $\mathbb{R}^n$
- Fact: null(A) is the set of vectors orthogonal to the rows of A

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix} \qquad Ax = \begin{bmatrix} a_1^T x \\ \vdots \\ a_n^T x \end{bmatrix} = 0$$

- Fact: rank(A) is the number of linearly independent rows of A

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix}$$

Basis for null(A)

$$A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 0, \quad A = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 0$$

Basis for null(A) is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$

# LinAlg Revision: Square Matrix Inverse

02 ODEs and Linear Algebra

**Definition**: The inverse of a matrix $A \in \mathbb{R}^{n\times n}$ is a matrix $A^{-1} \in \mathbb{R}^{n\times n}$

$$A^{-1}A = AA^{-1} = I$$

**Definition:** A matrix is called singular if it does not have an inverse. Otherwise it is called non-singular or invertible.

**Fact 2.9**: If an inverse of $A$ exists then it is unique.

**Fact 2.10**: $A$ is invertible if and only if $\det(A) \neq 0$

**Fact 2.11**: $A$ is invertible if and only if the system of linear equations $Ax = y$ has a unique solution $x \in \mathbb{R}^n$ for all $y \in \mathbb{R}^n$

**Fact 2.12**: $A$ is invertible if and only if $\text{null}(A) = \{0\}$

**Fact 2.13**: $A$ is invertible if and only if $\text{range}(A) = \mathbb{R}^n$

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So how do we compute the matrix inverse?

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$$

$n = 2$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{ad - bc}$$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \frac{1}{3 \cdot 2 - 0} \Rightarrow \boxed{A^{-1} = \begin{bmatrix} 1/2 & -1/6 \\ 0 & 1/3 \end{bmatrix}}$$

# LinAlg Revision: Eigenvalues and Eigenvectors

02 ODEs and Linear Algebra

**Definition**: A (nonzero) vector $w \in \mathbb{C}^n$ is called an eigenvector of a matrix $A \in \mathbb{R}^{n \times n}$ if there exists a number $\lambda \in \mathbb{C}$ such that $Aw = \lambda w$. The number $\lambda$ is then called an eigenvalue of $A$

**Fact 2.17**: $A$ is invertible if and only if all its eigenvalues are non-zero

An $n$x$n$ matrix has $n$ eigenvalues (some may be repeated). They are the solutions of the characteristic polynomial

$$\det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n = 0$$

The $n$ eigenvalues of $A$ are called the spectrum of $A$

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 \\ 2 & \lambda - 5 \end{vmatrix}$$

$$= \lambda^2 - 7\lambda + 12 = 0$$

$$\Rightarrow (\lambda - 4)(\lambda - 3)$$

$$\rightarrow \boxed{\lambda = 3, 4}$$

$$Aw = \lambda w$$

$$\lambda = 3, \quad Aw = 3w$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3w_1 \\ 3w_2 \end{bmatrix}$$

$$\Rightarrow 2w_1 + w_2 = 3w_1$$

$$\underline{\underline{w_2 = w_1}} \Rightarrow \boxed{w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

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$$A w = \lambda w$$

$$\lambda = 4, \quad A w = 4 w$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4w_1 \\ 4w_2 \end{bmatrix}$$

$$\Rightarrow 2w_1 + w_2 = 4w_1$$

$$\Rightarrow \underline{\underline{w_2 = 2w_1}}$$

$$\Rightarrow \boxed{w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

# LinAlg Revision: Symmetric positive (semi)definite matrices

02 ODEs and Linear Algebra

# Symmetric positive definite and positive semidefinite matrices

**Definition:** A matrix is called <u>symmetric</u> if $A = A^T$

$a_{ij} = a_{ji}$

**Definition:** A symmetric matrix is called <u>positive definite</u> if $x^T Ax > 0$ for all $x \neq 0$. It is called <u>positive semi-definite</u> if $x^T Ax \geq 0$.

- Fact: Symmetric matrices have real eigenvalues and orthogonal eigenvectors

- Fact: A symmetric matrix is positive definite (semidefinite) if and only if it has real positive (non-negative) eigenvalues

- Fact: If A is positive definite (semidefinite) there exists a matrix $A^{1/2} > 0$ ($A^{1/2} >= 0$) such that $A^{1/2}A^{1/2} = A$

- Notation: $A > 0$ $\qquad$ $A \succ 0$

# ODE Revision:<br>State Space Models

02 ODEs and Linear Algebra

# State Space Models: Inputs, Outputs, and States

- Mathematical model of physical system
  - input variables $u_1, u_2, \dots, u_m \in \mathbb{R}$
  - output variables $y_1, y_2, \dots, y_p \in \mathbb{R}$
  - state variables $x_1, x_2, \dots, x_n \in \mathbb{R}$
- Stack into vectors for compact notation $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^m, \; y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} \in \mathbb{R}^p, \; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$
- Number of states, *n*, called dimension or order of the system

# Dynamics

- System dynamics give relations between variables
  - Differential equations: evolution of states as a function of states, inputs, and possibly time

$$f_i(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \longrightarrow \mathbb{R} \qquad \frac{d}{dt} x_i(t) = f_i\big(x(t), u(t), t\big)$$

  - Algebraic equations: output as a function of states, inputs, and possibly time

$$h_i(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \longrightarrow \mathbb{R} \qquad y_i(t) = h_i\big(x(t), u(t), t\big)$$

- Often come from “laws of Nature”
  - Newton’s laws for mechanical systems
  - Electrical laws for circuits
  - Energy and mass balance for chemical systems

# State Space Models

- Again, stack into vectors for compact notation

$$f(\cdot): \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n \qquad h(\cdot): \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^p$$

$$f(x,u,t) = \begin{bmatrix} f_1(x,u,t) \\ \vdots \\ f_n(x,u,t) \end{bmatrix} \qquad h(x,u,t) = \begin{bmatrix} h_1(x,u,t) \\ \vdots \\ h_p(x,u,t) \end{bmatrix}$$

- State Space Form

$$\boxed{\begin{aligned} \frac{d}{dt}x(t) &= f(x(t), u(t), t) \\ y(t) &= h(x(t), u(t), t) \end{aligned}}$$

  - a system of coupled, first-order ODEs and algebraic equations
  - dynamics function sometimes called a vector field

# System Classifications

- Time invariant $\frac{d}{dt}x(t) = f(x(t), u(t))$ , $y(t) = h(x(t), u(t))$
- Autonomous $\frac{d}{dt}x(t) = f(x(t))$ , $y(t) = h(x(t))$
- Linear $\frac{d}{dt}x(t) = A(t)x(t) + B(t)u(t)$ $y(t) = C(t)x(t) + D(t)u(t)$
- Linear time invariant (LTI) $\frac{d}{dt}x(t) = Ax(t) + Bu(t)$ , $y(t) = Cx(t) + Du(t)$

# ODE Revision:
# Higher Order ODEs

02 ODEs and Linear Algebra

# Converting Higher Order ODEs to State Space Form

- Sometimes dynamics expressed in terms of higher order differential equations

$$\frac{d^n y(t)}{dt^n} + g_1\left(y(t), \frac{dy(t)}{dt}, \dots, \frac{d^{n-1} y(t)}{dt^{n-1}}\right) = g_2(u(t))$$

- Can always convert to state space form by defining state variables in terms of lower order derivatives

# Example: Linear ODE with an input

$$\underline{\frac{d^r}{dt^r} y(t)} + a_{r-1} \frac{d^{r-1}}{dt^{r-1}} y(t) + a_{r-2} \frac{d^{r-2} y(t)}{dt^{r-2}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = u(t)$$

Define state variable

$$x_1 = y(t) \implies \dot{x}_1 = \dot{y}(t) = x_2$$

$$\underline{x_2 = \dot{y}(t)} \implies \dot{x}_2 = \ddot{y}(t) = x_3$$

$$\vdots \qquad\qquad \vdots$$

$$x_r = \frac{d^{r-1}}{dt^{r-1}} y(t) \implies \dot{x}_r = \frac{d^r}{dt^r} y(t) = -a_{r-1} \underbrace{\frac{d^{r-1}}{dt^{r-1}} y(t)}_{x_r} - \dots - a_1 \underbrace{\frac{dy(t)}{dt}}_{x_2} - a_0 \underbrace{y(t)}_{x_1} + u(t)$$

# Example: Linear ODE with an input

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_r \end{bmatrix}
=
\underbrace{\begin{bmatrix}
0 & 1 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & & 0 & 1 & \ddots & \\
\vdots & & & & \ddots & \ddots \\
\vdots & & & & & \\
-a_0 & -a_1 & \cdots & \cdots & -a_{r-2} & -a_{r-1}
\end{bmatrix}}_{A}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_r \end{bmatrix}
+
\underbrace{\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B}
u(t)
$$

# ODE Revision: Solutions of State Space Equations

02 ODEs and Linear Algebra

# Solution of State Space Equations

- For now focus on autonomous time invariant systems

$$\dot{x}(t) = f(x(t)) \qquad y(t) = h(x(t))$$

- What is the solution of the system?
  - Given dynamics and output function, initial condition: $x(t_0) = x_0 \in \mathbb{R}^n$
  - What does the state and output do from now until some future time?

$$t_1 > t_0$$

- We want functions that "satisfy" the system equations and initial conditions

$$x(\cdot) : [t_0, t] \to \mathbb{R}^n \qquad y(\cdot) : [t_0, t_1] \to \mathbb{R}^p$$

# Solution definition

**Definition**: A pair of functions $x(\bullet):[t_0,t_1]\to\mathbb{R}^n, y(\bullet):[t_0,t_1]\to\mathbb{R}^p$ is a solution of the state space system over the interval $[t_0,t_1]$ starting at $x_0\in\mathbb{R}^n$ if

1. $x(t_0)=x_0$ 2. $\dot{x}(t)=f(x(t)), \quad \forall t\in[t_0,t_1]$
3. $y(t)=h(x(t)), \quad \forall t\in[t_0,t_1]$

- Hard part is finding state trajectory, just substitute to get output trajectory
- For autonomous systems, initial time unimportant

# Existence and Uniqueness Issues

- Does a solution exist for some time interval?
- Is the solution unique, or can there be more than one?
- Does the solution exist for arbitrary time intervals?
- Can the solution be computed? Analytically? ⟵ happens often!

  - Unfortunately, things can go wrong for all of these questions!
  - Problem is then with the *model*

# ODE Revision: Solutions of State Space Equations

02 ODEs and Linear Algebra

ate Transformations

are many equivalent ways to express dynamics

happens when we change coordinates? Why would we want to do

der linear time invariant (LTI) systems

$$\frac{d}{dt} x(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) + Du(t)$$

change of coordinates

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# Coordinate Transformations

- We will get another linear time invariant system:

$$\hat{x}(t) = T x(t) \iff \underline{x(t) = T^{-1}\hat{x}(t)}$$

$$\dot{\hat{x}}(t) = T\dot{x}(t) = T\left(A x(t) + B u(t)\right)$$

$$= \underbrace{TAT^{-1}}_{\tilde{A}}\hat{x}(t) + \underbrace{TB}_{\tilde{B}}u(t)$$

$$y(t) = C x(t) + \cancel{\ } D u(t)$$

$$y(t) = \underbrace{CT^{-1}}_{\tilde{C}}\hat{x}(t) + D u(t)$$

- Why? Transformed system may take some useful or simpler form with an interesting interpretation.

# Linear time invariant systems: solutions

03 Continuous LTI systems

# Linear time invariant systems

$$\boxed{\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}}$$

$x_0 \in \mathbb{R}^n$ $\quad T > 0$

$\underbrace{u(\bullet):[0,T] \to \mathbb{R}^m}_{\text{CONTINUOUS}}$

SOLUTION

$x(\cdot):[0,T] \to \mathbb{R}^n$
$y(\cdot):[0,T] \to \mathbb{R}^p$
S.T.

$$\checkmark \begin{cases} x(0) = x_0 \\ \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad \forall t \in [0,T]$$

- State transition matrix $\Phi(t) = e^{At} = \underbrace{I + At + \frac{A^2t^2}{2!} + \ldots + \frac{A^kt^k}{k!} + \ldots}\in \mathbb{R}^{n\times n}$

# Properties of the state transition matrix

**Fact:** The state transition matrix satisfies

WHY?

1. $\Phi(0) = I$
2. $\dfrac{d}{dt}\Phi(t) = A\Phi(t)$
3. $\Phi(-t) = [\Phi(t)]^{-1}$
4. $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)$

$$\Phi(t) = e^{At} = I + At + \ldots + \frac{A^k t^k}{k!} + \ldots$$

$$\phi(0) = I + \cancel{A \cdot 0}^{\,0} + \cdots + \cancel{\frac{A^k 0^k}{k!}}^{\,0} + \text{——} = I$$

$$\frac{d}{dt}\left[\cancel{I}^{\,0} + \cancel{At}^{\,A} + \text{——} + \cancel{\frac{A^k t^k}{k!}}^{\,\frac{A^k t^{k-1}}{(k-1)!}} + \text{——}\right]$$

$$= A\underbrace{\left[I + \cdots + \frac{A^{k-1}t^{k-1}}{(k-1)!} + \text{——}\right]}_{\phi(t)} = A\,\phi(t)$$

$$\phi(t)\cdot\phi(-t) = I \to \phi(0)\cdot\phi(-0) = I \cdot I = I$$

$$\frac{d}{dt}\left[\phi(t)\cdot\phi(-t)\right] = \cdots = 0$$

# Linear time invariant systems: Solution

- State solution

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau$$

- Output solution $y(t) = Cx(t) + Du(t)$

$$y(t) = C\Phi(t)x_0 + \int_0^t C\Phi(t-\tau)Bu(\tau)\,d\tau + Du(t)$$

# LTI systems: solution proof

03 Continuous LTI systems

# Linear time invariant systems: Solution proof

- Candidate solution $x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau$
- Show that it satisfies $x(0) = x_0$ $\longrightarrow x(0) = \cancel{\phi(0)}^{I} x_0 + \cancel{\int_0^0 \phi(0-\tau)Bu(\tau)\,d\tau}^{0} = I \cdot x_0 = x_0$ ✓

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \forall t \in [0, T]$$

- Leibniz rule

$$\frac{d}{dt}\int_{f(t)}^{g(t)} l(t,\tau)\,d\tau = l(t,g(t))\frac{d}{dt}g(t) - l(t,f(t))\frac{d}{dt}f(t) + \int_{f(t)}^{g(t)} \frac{\partial}{\partial t} l(t,\tau)\,d\tau$$

# Linear time invariant systems: Solution proof

$$\frac{d}{dt}\int_{f(t)}^{g(t)} l(t,\tau)\,d\tau = l(t,g(t))\frac{d}{dt}g(t) - l(t,f(t))\frac{d}{dt}f(t) + \int_{f(t)}^{g(t)}\frac{\partial}{\partial t}l(t,\tau)\,d\tau$$

- Differentiate candidate $\frac{d}{dt}x(t) = \underline{\frac{d}{dt}\Phi(t)x_0} + \underline{\frac{d}{dt}\int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau}$

$$\dot{x}(t) = \underbrace{\left(\frac{d}{dt}\phi(t)\right)}_{\underline{A\phi(t)}} x_0 + \underbrace{\phi(t-t)}_{I} Bu(t)\cdot\cancel{\frac{d}{dt}(t)} - \underline{\phi(t-0)}\,Bu(0)\cdot\frac{d}{dt}(0) + \int_0^t \underbrace{\frac{\partial}{\partial t}\phi(t-\tau)}_{\underline{A\phi(t-\tau)}} Bu(\tau)\,d\tau$$

$$\frac{d}{dt}(t) \rightarrow 1 \qquad \frac{d}{dt}(0) \rightarrow 0$$

$$\dot{x}(t) = A\underbrace{\left[\phi(t)x_0 + \int_0^t \phi(t-\tau)Bu(\tau)\,d\tau\right]}_{x(t)} + Bu(t)$$

$$\boxed{\dot{x}(t) = Ax(t) + Bu(t)} \quad \forall t \in [0,T]$$

# Linear time invariant systems: Solution

LINEAR IN $x_0 \in \mathbb{R}^n$

LINEAR IN $u(t)$

" SUPERPOSITION PRINCIPLE "

- State solution

$$\underbrace{\boxed{x(t)}}_{} = \underbrace{\boxed{\Phi(t)x_0}}_{u(t)=0} + \underbrace{\boxed{\int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau}}_{x_0=0}$$

Total transition = Zero Input transition + Zero State transition

- Output solution

$$\underbrace{\boxed{y(t)}}_{} = \underbrace{\boxed{C\Phi(t)x_0}}_{u(t)=0} + \underbrace{\boxed{C\int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau + Du(t)}}_{x_0=0}$$

Total Response = Zero Input Response + Zero State Response

# LTI systems:
# Computing the state transition matrix

03 Continuous LTI systems

# Linear time invariant systems: Solution

$$\longrightarrow \quad x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau$$

- Key ingredient: State transition matrix

$$\boxed{\Phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \ldots + \frac{A^k t^k}{k!} + \ldots} \in \mathbb{R}^{n\times n}$$

- Compute based on eigenvalue/eigenvector decomposition

# Diagonalizable matrices

- Eigenvectors: $w_i \in \mathbb{C}^n, w_i \neq 0 : Aw_i = \lambda_i w_i$ for some $\lambda_i \in \mathbb{C}, i = 1,\dots,n$
- Matrix diagonalizable if eigenvectors linearly independent
- Matrix $W = \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix} \in \mathbb{C}^{n\times n}$ is invertible

$$A \cdot W = A[w_1 \ \dots \ w_n] = [Aw_1 \ \dots \ Aw_n] = [\lambda_1 w_1 \ \dots \ \lambda_n w_n] = \underbrace{[w_1 \ \dots \ w_n]}_{W} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda \in \mathbb{C}^{n\times n}}$$

$\underbrace{\det[\lambda I - A]}_{\text{REAL COEFS}} = 0 \rightsquigarrow$ ROOTS $\lambda_i \in \mathbb{C}, w_i \in \mathbb{C}^n$

COMPLEX CONJUGATE

COMPLEX $\mathbb{C}^{n\times n}$

REAL $\mathbb{R}^{n\times n}$

$$\boxed{A = W \Lambda W^{-1}}$$

# State transition matrix computation: Diagonalizable matrices

$$A = W\Lambda W^{-1} \longrightarrow A^2 = (W\Lambda \underbrace{W^{-1})(W}_{I}\Lambda W^{-1}) = W\Lambda^2 W^{-1}, \dots \quad A^k = W\Lambda^k W^{-1}$$

- Substitute into the state transition matrix

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots + \frac{A^kt^k}{k!} + \dots$$

$$= W \cdot W^{-1} + W\Lambda t W^{-1} + \dots + \frac{W\Lambda^k t^k W^{-1}}{k!} + \dots$$

$$= W\left[I + \Lambda t + \dots + \frac{\Lambda^k t^k}{k!} + \dots\right] W^{-1}$$

$\mathbb{C}^{n\times n}$

LINEAR COMBINATIONS $e^{\lambda_i t}$

$\mathbb{R}^{n\times n}$

$$\boxed{e^{At} = We^{\Lambda t}W^{-1}}$$

# State transition matrix computation: Diagonalizable matrices

$$e^{\Lambda t} = \begin{bmatrix} 1+\lambda_1 t+\ldots+\frac{\lambda_1^k t^k}{k!}+\ldots & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1+\lambda_n t+\ldots+\frac{\lambda_n^k t^k}{k!}+\ldots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{\lambda_n t} \end{bmatrix}$$

LINEAR COMBOS OF $e^{\lambda_1 t}$ — $e^{\lambda_n t}$

$$\boxed{e^{At} = W \begin{bmatrix} e^{\lambda_1 t} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{\lambda_n t} \end{bmatrix} W^{-1}}$$

# LTI systems: Structure of the solutions

03 Continuous LTI systems

# Solution structure $e^{At} = We^{\Lambda t}W^{-1}$

- Linear combination of $e^{\lambda_i t}$

$$\lambda_i = \sigma \pm j\omega$$

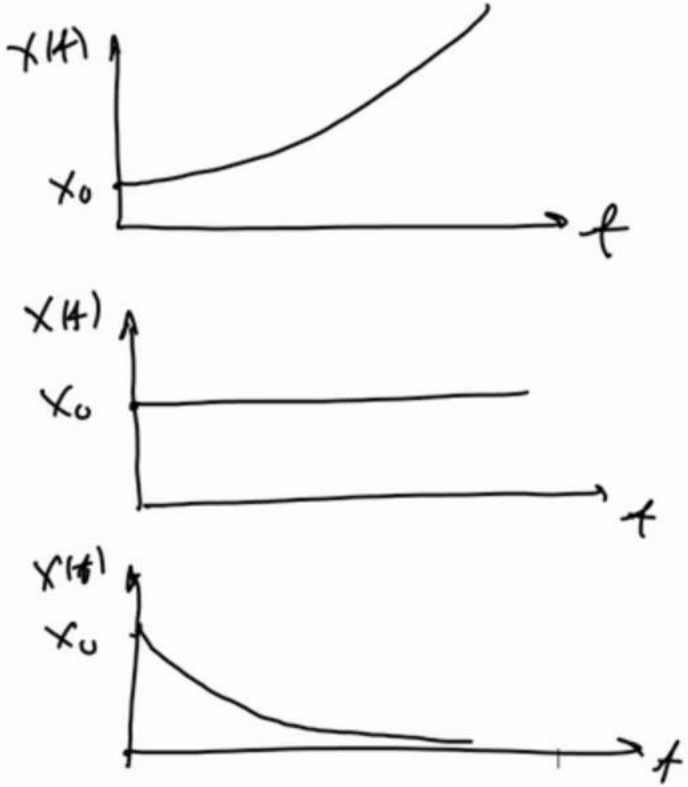
$$e^{\lambda_i t} = \underbrace{e^{\sigma t}}_{\mathbb{R}} \cdot \underbrace{e^{\pm j\omega t}}_{\cos\omega t \pm j\sin\omega t}$$

$$|e^{\lambda_i t}| = e^{\sigma t} \cdot \cancel{|e^{\pm j\omega t}|} \rightarrow 1$$

$$\angle e^{\lambda_i t} = \tan^{-1}(\omega t) \rightarrow \omega = 0 \quad \mathbb{R}$$

- $\omega = 0 \Rightarrow \lambda_i = \sigma$

$$\begin{cases} \sigma > 0 \Rightarrow e^{\lambda_i t} \rightarrow \infty \\ \sigma = 0 \Rightarrow e^{\lambda_i t} = 1 \\ \sigma < 0 \Rightarrow e^{\lambda_i t} \rightarrow 0 \end{cases}$$

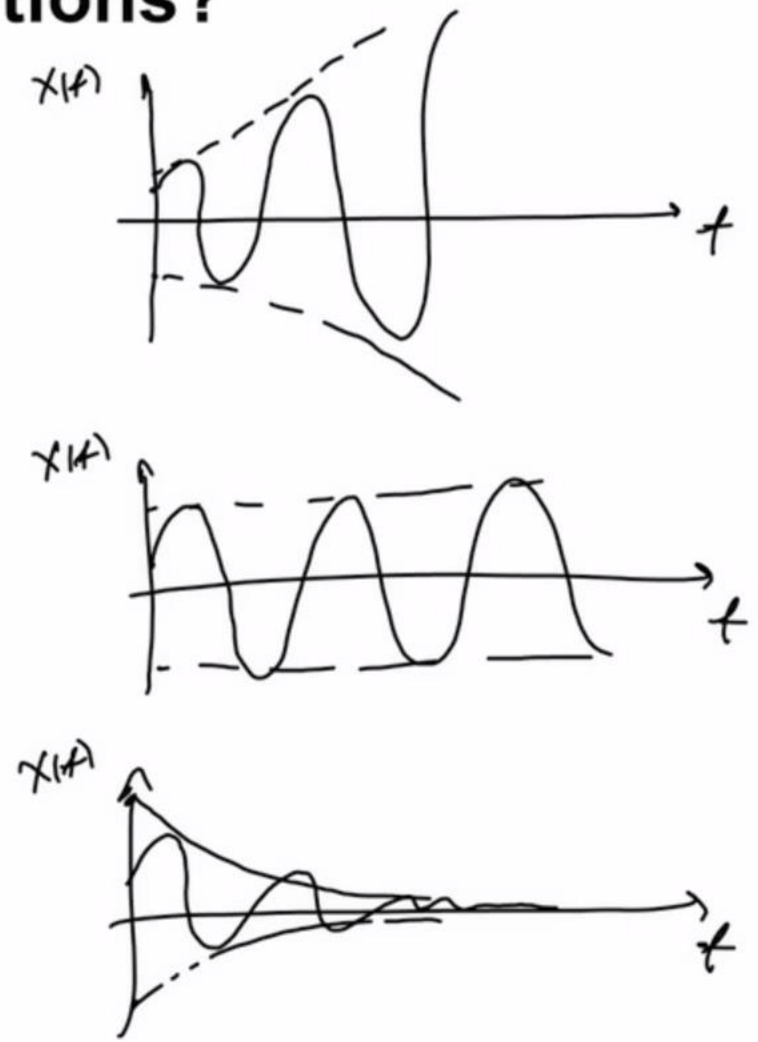


# What does this mean about the solutions?

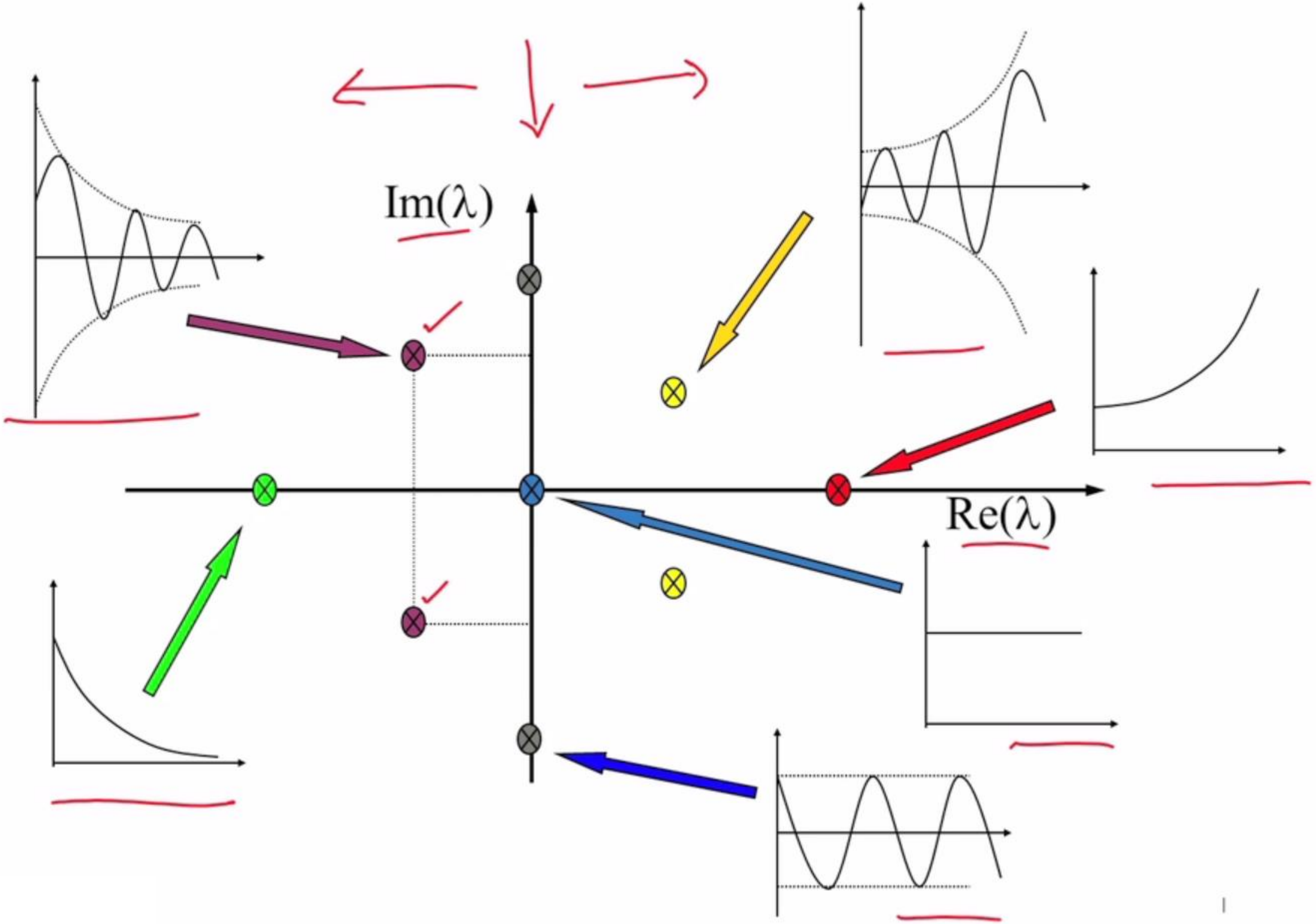
- $$\omega \neq 0 \quad \begin{cases} \sigma > 0 \Rightarrow \left| e^{\lambda_i t} \right| \to \infty \\ \sigma = 0 \Rightarrow \left| e^{\lambda_i t} \right| = 1 \\ \sigma < 0 \Rightarrow \left| e^{\lambda_i t} \right| \to 0 \end{cases}$$

cos ωt, sin ωt

OSCILLATING



# Summary



# LTI systems: stability definitions

03 Continuous LTI systems

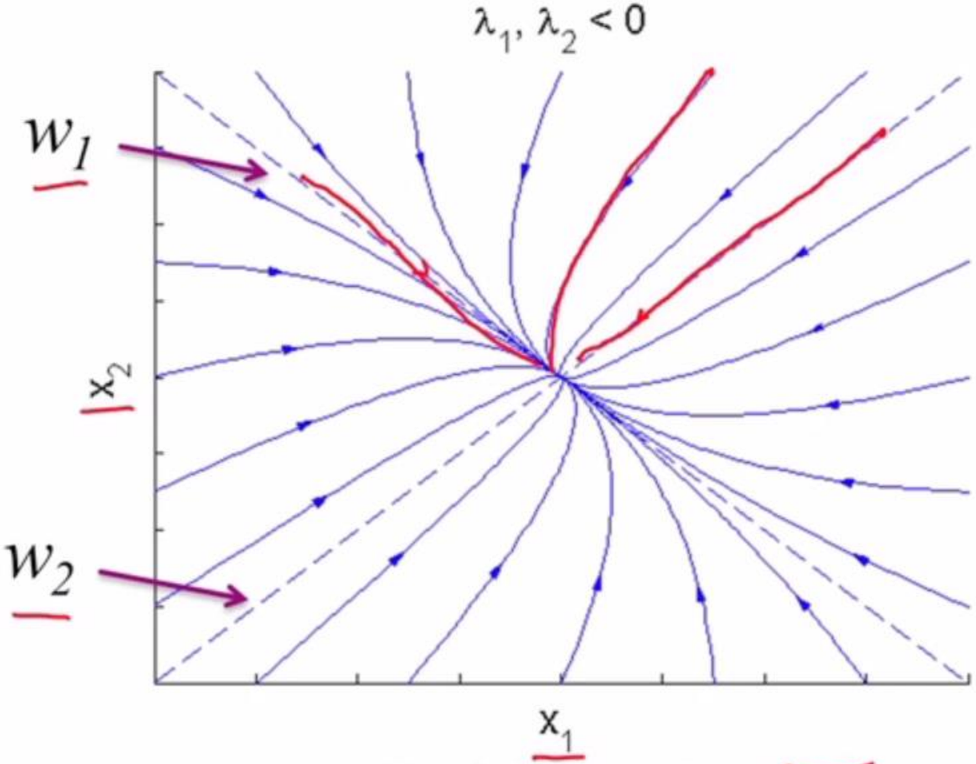
# State transition matrix: Diagonalizable matrices

$$\boxed{e^{At} = W \begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n t} \end{bmatrix} W^{-1}}$$

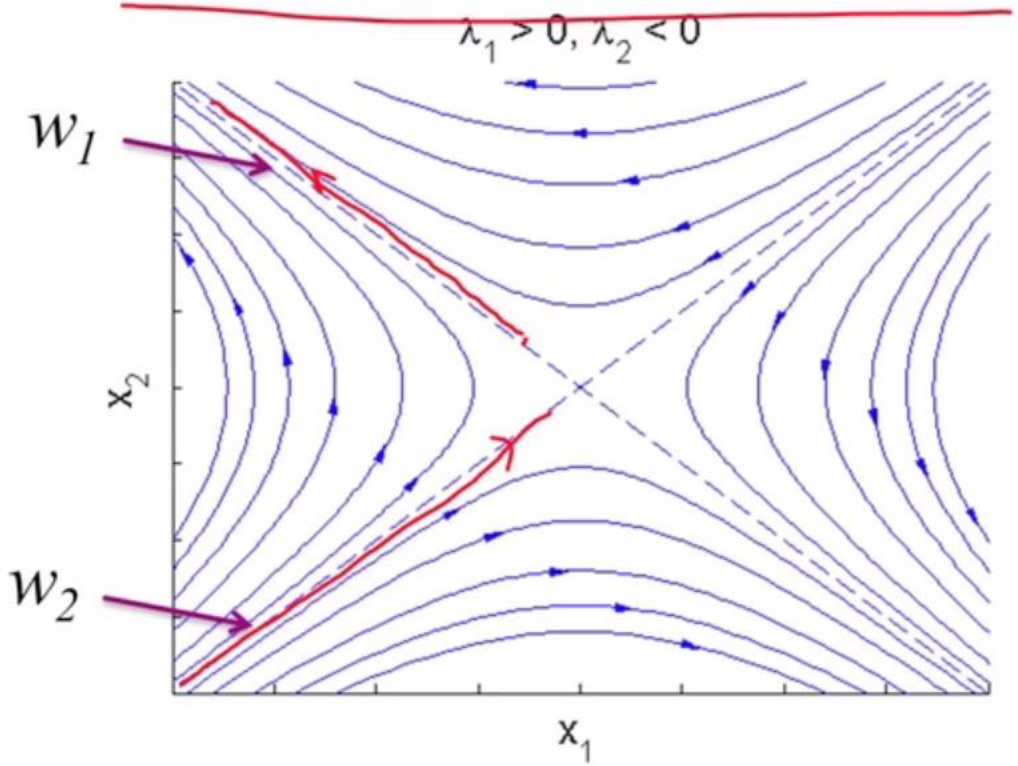
- Zero input transition $x(t) = e^{At} x_0$

**Phase plane plots** $x(t) \in \mathbb{R}^2$

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) \\ x(0) &= x_0 \end{aligned}\right\} \Rightarrow x(t) = W \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} \end{bmatrix} W^{-1} x_0$$



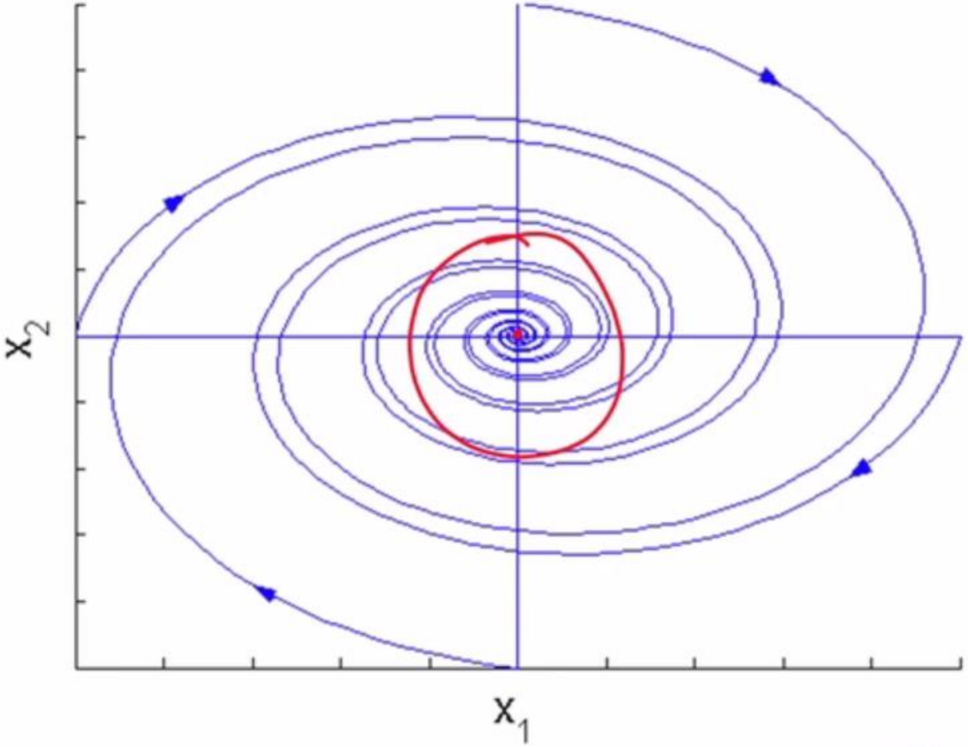

Stable node




Saddle point

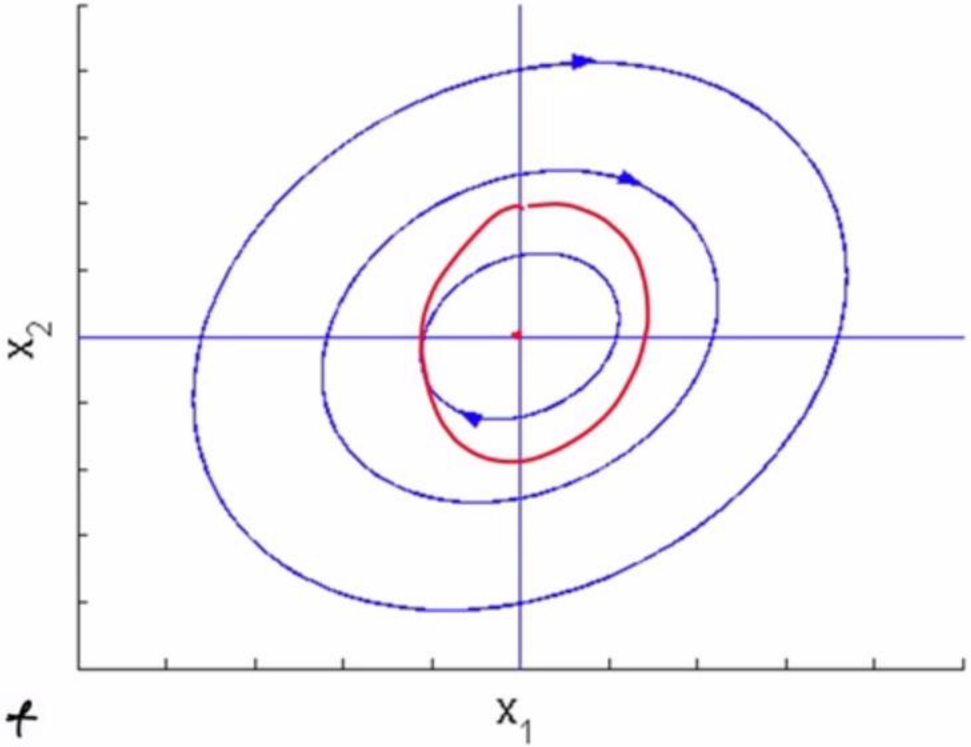
# Phase plane plots: Complex eigenvalues

$\lambda_1 = \sigma + j\omega,\ \lambda_2 = \sigma - j\omega;\ \sigma < 0$




Stable focus

$\lambda_1 = +j\omega,\ \lambda_2 = -j\omega$ $\sigma = 0$




Center

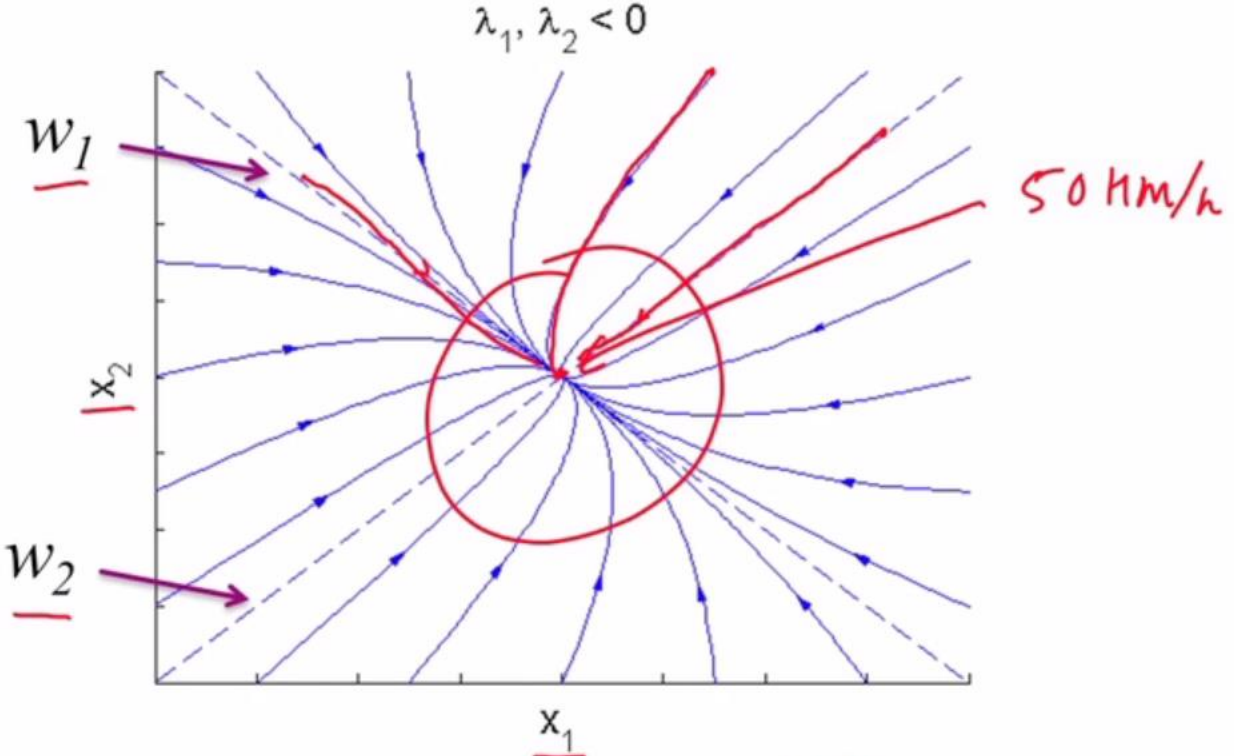
$X_0 = 0 \Rightarrow X(t) = 0\ \forall t$

$\dot{X}(t) = AX(t) = 0$

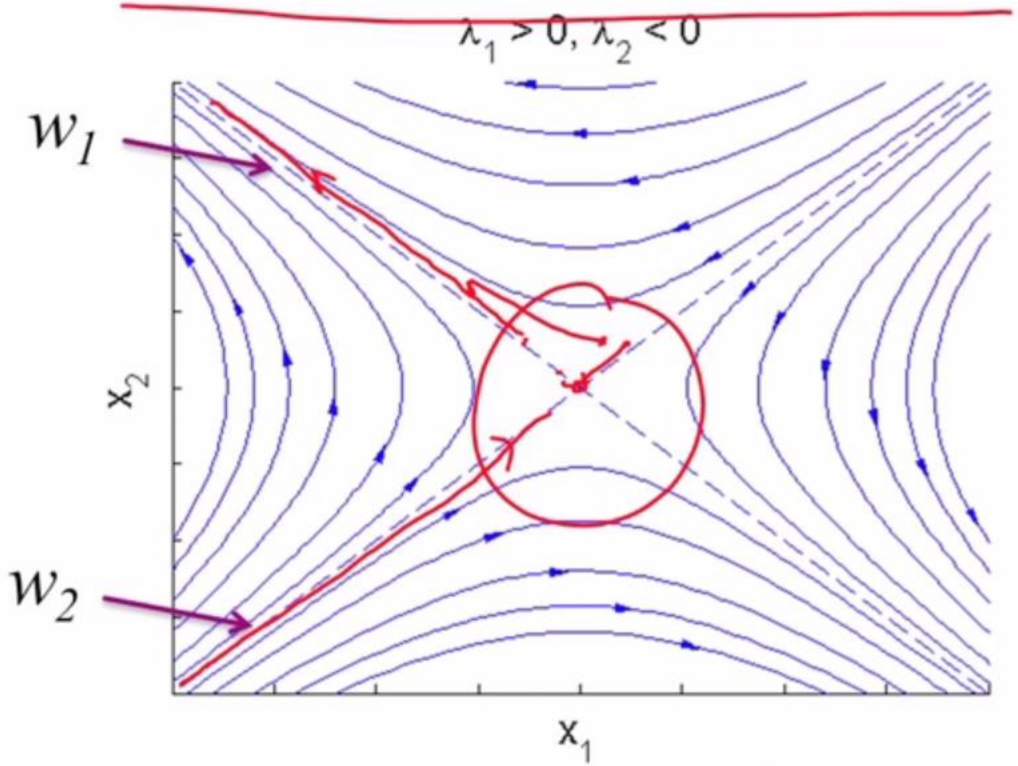
EQUILIBRIUM

**Phase plane plots**
$x(t) \in \mathbb{R}^2$

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) \\ x(0) &= x_0 \end{aligned}\right\} \Rightarrow x(t) = W \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} \end{bmatrix} W^{-1} x_0$$




Stable node




Saddle point

## Stability definitions

- Zero state transition $x(t) = \Phi(t)x_0 = e^{At}x_0$

**Definition**: The system is called stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } \|x_0\| \leq \delta \text{ then } \|x(t)\| \leq \varepsilon \text{ for all } t \geq 0.$$

Otherwise the system is called unstable.

**Definition**: The system is called asymptotically stable if it is stable and in addition

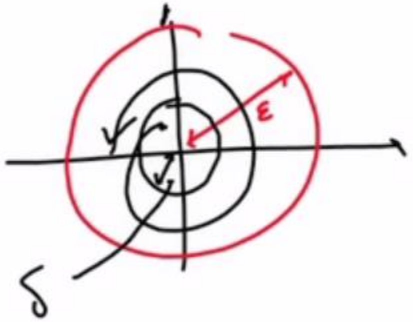
$$\|x(t)\| \to 0 \text{ as } t \to \infty$$

# LTI systems: stability conditions

03 Continuous LTI systems

# Stability conditions: Diagonalizable matrices

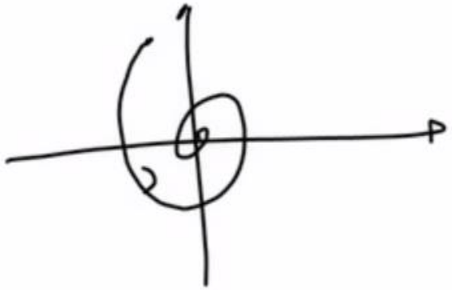
STABLE
"START SMALL
STAY SMALL"
P1 $\to \varepsilon$
P2 $\to \delta$




$\Longleftarrow$

ASYMPTOTICALLY
STABLE

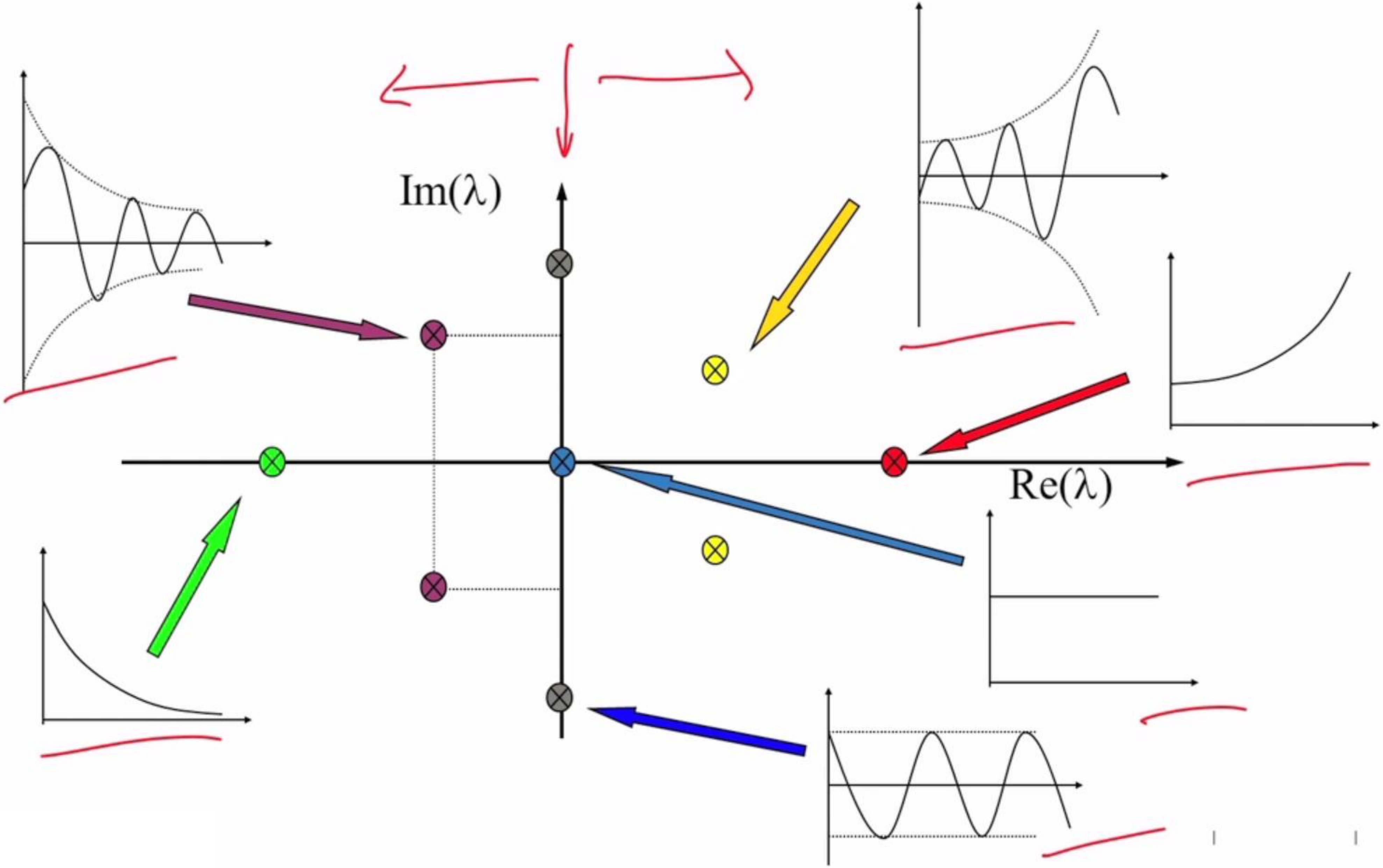
CONVERGE
TO 0



**Theorem 3.1**: System with diagonalizable $A$ matrix is:

- Stable if and only if $\mathrm{Re}[\lambda_i] \leq 0, \forall i$
- Asymptotically stable if and only if $\mathrm{Re}[\lambda_i] < 0 \ \forall i$
- Unstable if and only if $\exists i : \mathrm{Re}[\lambda_i] > 0$

# Intuition



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# Non-diagonalizable matrices

- Repeated eigenvalues $\lambda_1 = \lambda_2 = \ldots = \lambda_r = \sigma \pm j\omega$
- State transition matrix contains terms of the form $e^{\lambda_i t}, te^{\lambda_i t}, \ldots, t^{r-1}e^{\lambda_i t}$
- $\sigma > 0 \Rightarrow \left|t^k e^{\lambda_i t}\right| \to \infty$ UNSTABLE
- $\sigma = 0$
  - $r=1$, $|e^{\lambda_i t}| =$ CONSTANT. STABLE
  - $r>1$ $te^{\lambda_i t}$, $\cancel{te}\, t^2 e^{\lambda_i t}$, —
    - $t, t^2,$ — $\omega = 0$
    - $t\cos\omega t, t\sin\omega t$ — $\omega \neq 0$
- $\sigma < 0 \Rightarrow \left|t^k e^{\lambda_i t}\right| \to 0$ ASYMPTOTIC STABILITY


UNSTABLE.

# Stability conditions: Non-diagonalizable matrices

**Theorem 3.2**: The system is:

- Asymptotically stable if and only if $\mathrm{Re}[\lambda_i] < 0 \ \forall i$
- Unstable <u>if</u> $\exists i : \mathrm{Re}[\lambda_i] > 0$

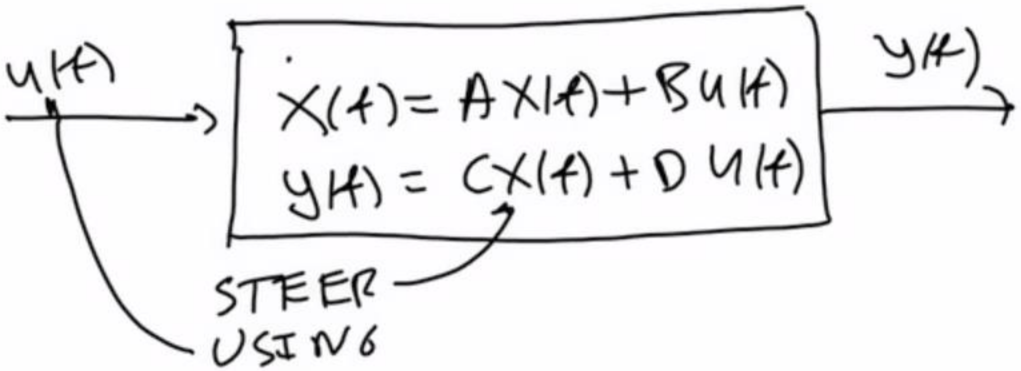
STABILITY ?

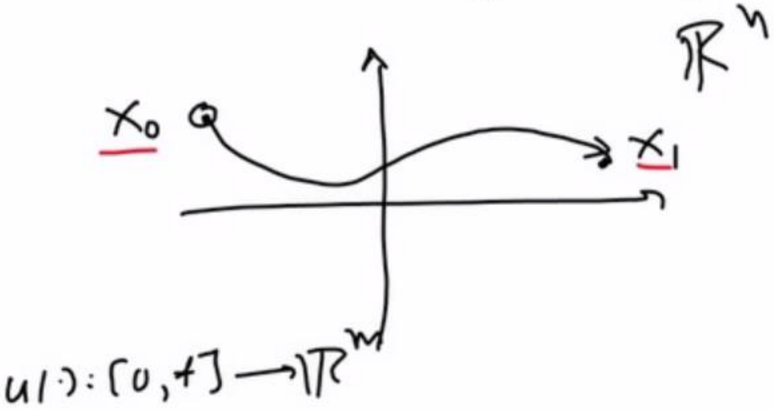
# Controllability: basic defintions

04 Energy Controllability Observability

# Controllability: Basic notion

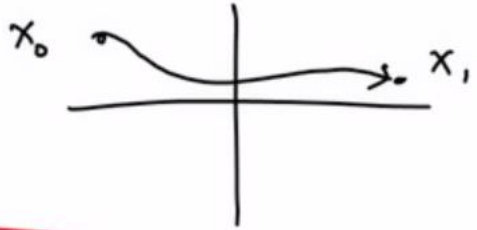
- Steer the state from where it is to where you want it to be using the input



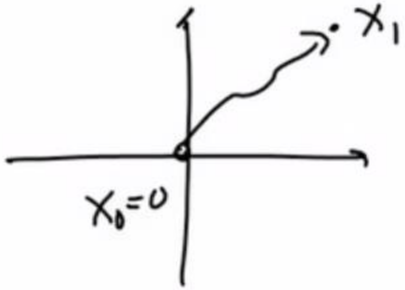




**Definition**: The system is called controllable over $[0, t]$ if for all $x(0) = x_0 \in \mathbb{R}^n$ initial conditions and all terminal $x_1 \in \mathbb{R}^n$ conditions there exists an input $u(\cdot) : [0, t] \to \mathbb{R}^m$ such that $x(t) = x_1$

# Controllability: Equivalent notions

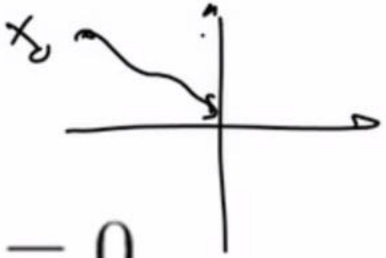



$$\forall x_0, x_1 \ \exists u(\cdot) : [0,t] \to \mathbb{R}^m : \quad \boxed{x_1 = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}$$




$$\forall x_1 \ \exists u(\cdot) : [0,t] \to \mathbb{R}^m : \quad x_1 = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

REACHABILITY




$$\forall x_0 \ \exists u(\cdot) : [0,t] \to \mathbb{R}^m : \quad e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = 0$$

# Controllability Gramian

$$W_C(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau \in \mathbb{R}^{n \times n}$$

$e^{A\tau} \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $B^T \in \mathbb{R}^{m\times n}$, $e^{A^T\tau} \in \mathbb{R}^{n\times n}$

FACT: $W_c(t) = W_c(t)^T \geq 0$ POSITIVE SEMI-DEF.

$\Rightarrow$ E-VALUES REAL $\geq 0$

$\Rightarrow x^T W_c(t) x \geq 0 \quad \forall x \in \mathbb{R}^n$

$W_c(t)$ INVERTIBLE $\Rightarrow$ E-VALUES $\neq 0$

$\Downarrow$

$W_c(t) > 0$ (E-VALUES $> 0$)

**Theorem**: The system is controllable over $[0, t]$ if and only if $W_C(t)$ is invertible

# Controllability matrix

$$P = [\underbrace{B}_{\mathbb{R}^{n\times m}} \quad \underbrace{AB}_{\mathbb{R}^{n\times n}} \quad A^2B \cdots A^{n-1}B] \in \mathbb{R}^{n\times(n\cdot m)}$$

$m = 1$
SINGLE INPUT

$P \in \mathbb{R}^{n\times n}$
CONTROLLABLE
$\Updownarrow$
P INVERTIBLE.
$\det[P] \neq 0$

$\mathrm{RANGE}[P] = \mathrm{RANGE}[W_c(t)]$

$W_c(t)$ INVERTIBLE
$\Updownarrow$
$\mathrm{RANGE}(W_c(t)) = \mathbb{R}^n$
$\Updownarrow$
$\mathrm{RANGE}[P] = \mathbb{R}^n$
$\Updownarrow$
$\mathrm{RANK}[P] = n$

# LINEARLY INDEP. COLUMNS OF P

**Theorem**: The system is controllable over $[0, t]$ if and only if the rank of $P$ is $n$

# Controllability: minimum energy controls

04 Energy Controllability Observability

# Summary of controllability definition and conditions

$$\forall x_0, x_1 \ \exists u(\cdot) : [0,t] \to \mathbb{R}^m : \quad x_1 = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

$$\Updownarrow$$

$$\forall x_1 \ \exists u(\cdot) : [0,t] \to \mathbb{R}^m : \quad x_1 = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

REACHABILITY

$$\Updownarrow$$

$$\int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau \ \text{ invertible} \iff \text{Rank}\left[B \ AB \ \ldots \ A^{n-1}B\right] = n$$

# Reachability

$$\boxed{\forall x_1 \; \exists u(\cdot) : [0,t] \to \mathbb{R}^m : \quad x_1 = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau}$$

"$X_1$ REACHABLE" IF $u(\cdot)$ EXISTS $X(0)=0 \rightsquigarrow X(t)=X_1$

$$\boxed{x_1 \in \textsc{Range}\, \overbrace{\left[B \; AB \; \dots \; A^{n-1}B\right]}^{P}}$$

SYSTEM CONTROLLABLE

RANK $[P] = n \iff$ RANGE $[P] = \mathbb{R}^n$

$\iff$ ALL $X_1 \in \mathbb{R}^n$ REACHABLE.

SYSTEM NOT CONTROLLABLE

RANK $[P] < n$

$\Downarrow$

RANGE$[P] \subsetneq \mathbb{R}^n$

REACHABLE STATES, SUBSPACE OF $\mathbb{R}^n$

# Minimum energy controls

$$\boxed{\int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau \ \text{ invertible for some } t > 0}$$

$$\Updownarrow$$

RANK [P] = n NOT INVOLVE t

$$\Updownarrow$$

$$\boxed{\int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau \ \text{ invertible for all } t > 0}$$

## Minimum energy controls

$$u(\cdot) : [0, t] \to \mathbb{R}^m, \quad \boxed{\text{“energy”} = \int_0^t \|u(\tau)\|^2 d\tau}$$

**Theorem**: Assume that the system is controllable. Given $x_1 \in \mathbb{R}^n$ and $t > 0$, the input that drives the system from $x(0)=0$ to $x(t)=x_1$ and has the minimum energy is given by

$$u_m(\tau) = B^T e^{A^T(t-\tau)} W_C(t)^{-1} x_1, \quad \text{for } \tau \in [0,t]$$

ENERGY IN $u_m(\cdot)$

$\parallel$

$x_1^T [W_c(t)]^{-1} x_1$

FURTHER $\Rightarrow$ MORE ENERGY.

$W_c(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau > 0$

$t_1 \le t_2 \longrightarrow W_c(t_2) - W_c(t_1) = \int_{t_1}^{t_2} e^{A\tau} BB^T e^{A^T\tau} d\tau \geq 0 \Rightarrow W_c(t_2) \geq W_c(t_1)$

$\Rightarrow [W_c(t_2)]^{-1} \le [W_c(t_1)]^{-1}$

FASTER

$\Downarrow$

MORE ENERGY.

# Observability: basic definitions

04 Energy Controllability Observability

# Observability: Definition

- Infer the value of the state by looking at the input and the output

$u(t)$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$y(t)$

MEASURE

INFER

**Definition**: The system is called observable over $[0, t]$ if given $u(\cdot):[0,t]\rightarrow\mathbb{R}^m$ and $y(\cdot):[0,t]\rightarrow\mathbb{R}^p$ we can uniquely determine the value of $x(\cdot):[0,t]\rightarrow\mathbb{R}^n$

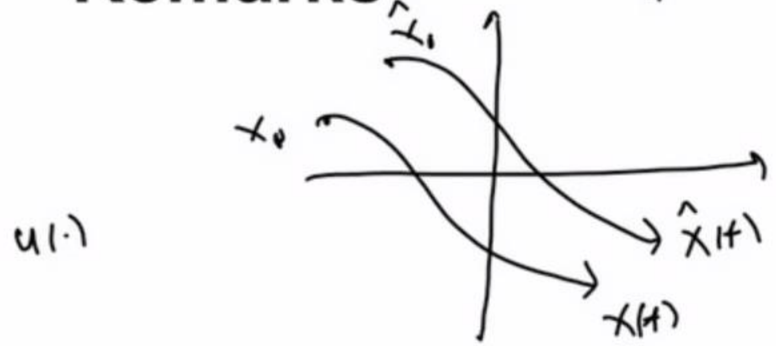
$x_0 \in \mathbb{R}^n$ ENOUGH TO DETERMINE $x(\cdot):[0,t]\rightarrow\mathbb{R}^n$

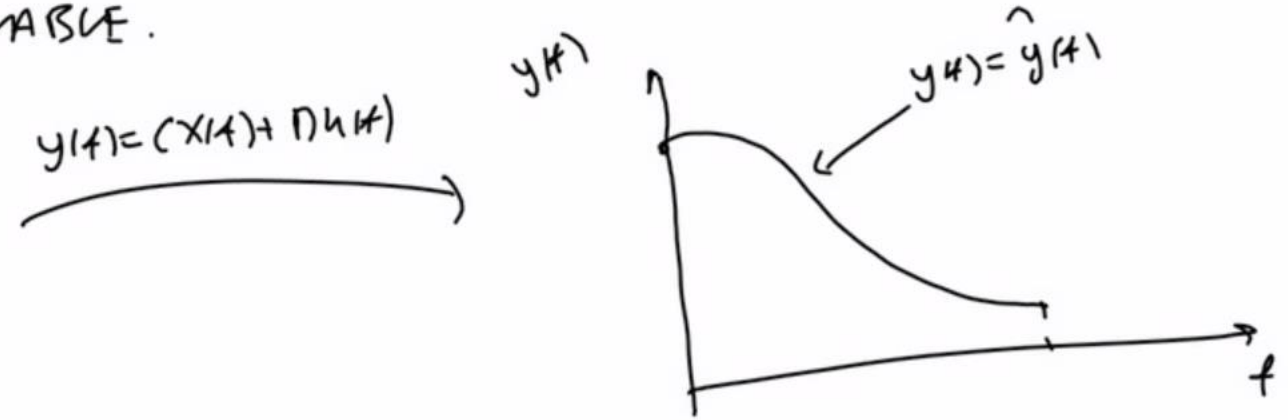
$$x(t) = e^{A\tau}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

UNOBSERVABLE.

$\mathbb{R}^n$

# Remarks




$$\left.\begin{aligned} y(\tau) &= \underline{C}e^{A\tau}x_0 + \int_0^t \cancel{Ce^{A(t-\tau)}}Bu(\tau)d\tau + \cancel{Du}(\tau) \\ \hat{y}(\tau) &= \underline{C}e^{A\tau}\hat{x}_0 + \int_0^t \cancel{Ce^{A(t-\tau)}}Bu(\tau)d\tau + \cancel{Du}(\tau) \end{aligned}\right\} \forall \tau \in [0,t]$$

$$y(\tau) - \hat{y}(\tau) = Ce^{A\tau}\underbrace{(x_0 - \hat{x}_0)}_{\text{"UNOBSERVABLE}} = 0 \quad \forall \tau \in [0,t]$$

ETH zürich

# Unobservable states

$x \in \mathbb{R}^n$ UNOBSERVABLE $\Leftrightarrow$ $Ce^{A\tau}x = 0 \quad \forall \tau \in [0,t]$

$x=0$ ALWAYS UNOBSERVABLE.

$\exists ?\ x \neq 0$ UNOBSERVABLE $\rightsquigarrow$ $x=0$ ONLY UNOBSERVABLE STATE
$\Leftrightarrow$ SYSTEM OBSERVABLE.

$y(\tau) = Ce^{A\tau}x = 0 \quad \underline{\forall \tau \in [0,t]}$

$$\left.\begin{aligned}
y(0) &= Ce^{A\cdot 0}x = Cx = 0 \\
\dot{y}(0) &= CAe^{A\cdot 0}x = CAx = 0 \\
\ddot{y}(0) &= \dots = CA^2x = 0 \\
&\ \vdots \\
y^{(k)}(0) &= CA^kx = 0
\end{aligned}\right\}$$

$x$ UNOBSERVABLE $\Leftrightarrow$ $\underline{CA^kx = 0}$ $\forall k = 0, 1, \dots, n-1$

ENOUGH TO GO TO $k = n-1$

$A^n = \alpha_1 A^{n-1} + \alpha_2 A^{n-2} \dots + \alpha_n I$

(CAYLEY HAMILTON)

# Observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \left. \begin{array}{l} \leftarrow \mathbb{R}^{p\times n} \\ \leftarrow \mathbb{R}^{p\times n} \\ \vdots \\ \leftarrow \mathbb{R}^{p\times n} \end{array} \right\} \mathbb{R}^{(p\cdot n)\times n}$$

**Theorem**: Set of unobservable states equal to Null($Q$)

**Theorem**: The system is observable over $[0, t]$ if and only if the rank of the matrix $Q$ is $n$

$P = 1$
SINGLE OUTPUT

$\text{RANK}(Q) = n \iff Q$ INVERTIBLE
$\text{DET}(Q) \neq 0$

# Initial state estimation

04 Energy Controllability Observability

ETH zürich

## Observability condition

$$Q \in \mathbb{R}^{p \cdot n \times n}$$

No t!

- System observable on [0, t] if and only if $\mathrm{RANK} \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q \in \mathbb{R}^{p \cdot n \times n}} = n$

t SMALL → ENOUGH TO RECONSTRUCT STATE.

- Observable for some t if and only if observable for all t!
- Consider differentiating y(t) along solutions of $\dot{x}(t) = Ax(t) + Bu(t)$

$$\left.\begin{aligned} y(t) &= Cx(t) + Du(t) \\ \dot{y}(t) &= C\dot{x}(t) + D\dot{u}(t) = CAx(t) + CBu(t) + D\dot{u}(t) \\ \ddot{y}(t) &= CA^2x(t) + CABu(t) + CB\dot{u}(t) + D\ddot{u}(t) \end{aligned}\right\} t=0$$

USE $\{u(t), y(t) \mid t \in [0, \varepsilon]\}$ → COMPUTE $y(0), \dot{y}(0), \ddot{y}(0)$ – –

$u(0), \dot{u}(0), \ddot{u}(0)$ ——

# Reconstructing the state using derivatives

FIND X(0)

$$\underbrace{\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix}}_{Y \in \mathbb{R}^{p \cdot n}} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q \in \mathbb{R}^{(p \cdot n) \times n}} \underbrace{x(0)}_{\mathbb{R}^n} + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ CA^{n-2}B & CA^{n-3}B & \cdots & D \end{bmatrix}}_{K \in \mathbb{R}^{(p \cdot n) \times (n \cdot m)}} \underbrace{\begin{bmatrix} u(0) \\ \dot{u}(0) \\ \vdots \\ u^{(n-1)}(0) \end{bmatrix}}_{\bar{U} \in \mathbb{R}^{n \cdot m}}$$

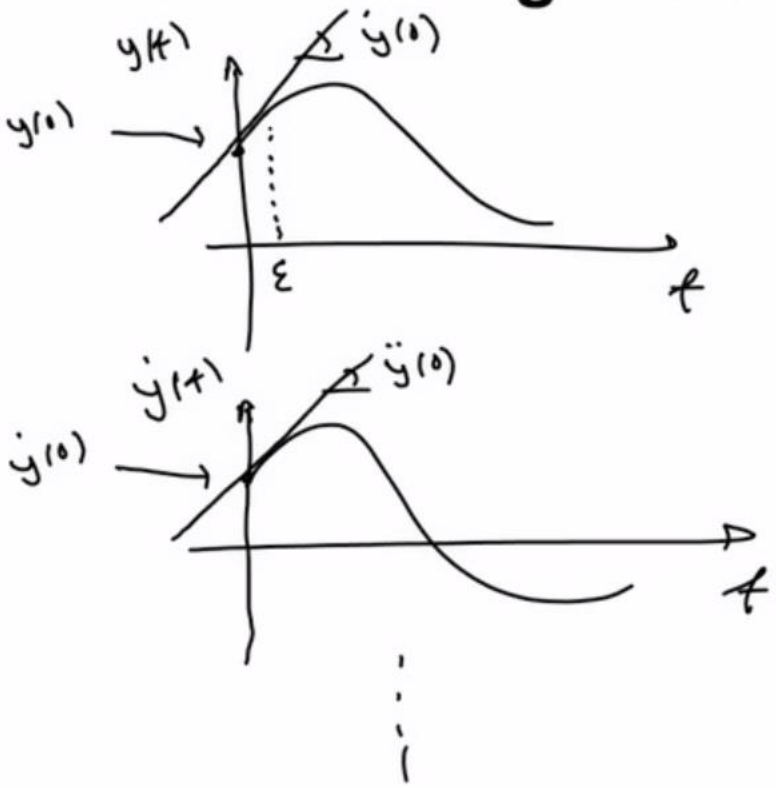
$\underset{\text{KNOWN}}{Y} = Q \underset{\text{UNKNOWN}}{X(0)} + K \underset{\text{KNOWN}}{\bar{U}}$ $\longrightarrow$ SOLVE FOR X(0) OBSERVABLE SYSTEM

RANK $(Q) = n \Rightarrow X(0) = \underbrace{(Q^T Q)^{-1} Q^T}_{\text{PSEUDO-INVERSE OF } Q} (Y - KU)$

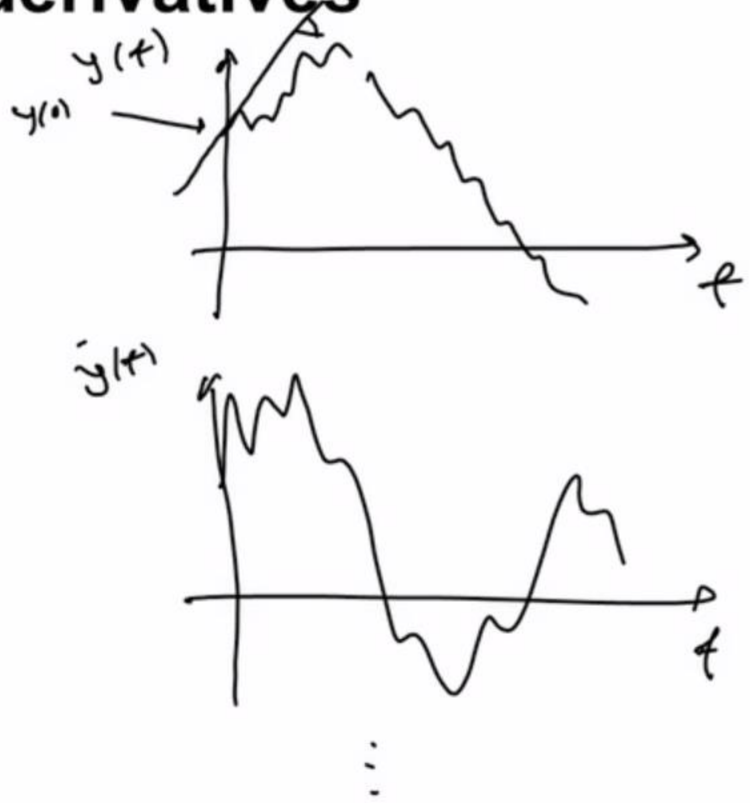
$\underset{\text{SINGLE OUTPUT}}{P = 1}$ $\quad Q \in \mathbb{R}^{n \times n} \longrightarrow$ RANK $(Q) = n \Leftrightarrow Q$ INVERTIBLE

$X(0) = Q^{-1}(Y - K\bar{U})$

# Reconstructing the state using derivatives




IN PRINCIPLE




IN PRACTICE.

DIFFERENTIATION AMPLIFIES NOISE!

# Observers

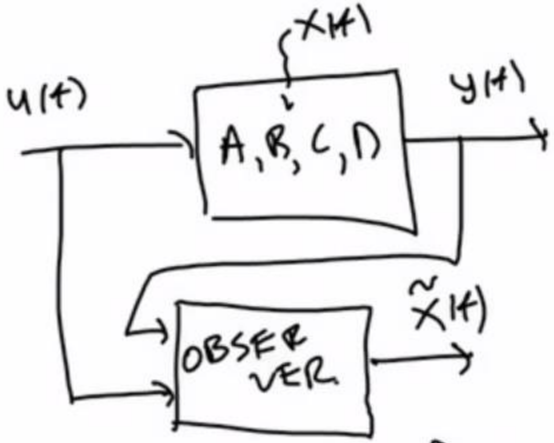
04 Energy Controllability Observability

# Observers

STATE ESTIMATE.

- Recursively construct estimate $\tilde{x}(t) \in \mathbb{R}^n$

REAL WORLD.




$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$L \in \mathbb{R}^{n \times p}$

GAIN MATRIX

CORRECTION TERM

IN COMPUTER

$$\dot{\tilde{x}}(t) = \underline{A\tilde{x}(t) + Bu(t)} + \overbrace{L(\underbrace{y(t) - \tilde{y}(t)}_{\text{OUTPUT ERROR}})}^{\text{CORRECTION TERM}}$$

$$\tilde{y}(t) = C\tilde{x}(t) + Du(t)$$

OUPUT ESTIMATE. STATE ESTIMATE. TRUE INPUT. REAL OUTPUT OUTPUT ERROR.

# Observation error evolution

- Observation error $e(t) = \underbrace{x(t) - \tilde{x}(t)}\in \mathbb{R}^n$ OBSERVATION ERROR.

$$\dot{e}(t) = \dot{x}(t) - \dot{\tilde{x}}(t) = Ax(t) + \cancel{Bu(t)} - A\tilde{x}(t) - \cancel{Bu(t)} - L(y(t) - \tilde{y}(t))$$

$$= A(x(t) - \tilde{x}(t)) - L(Cx(t) + \cancel{Du(t)} - C\tilde{x}(t) - \cancel{Du(t)})$$

$$= (A - LC)\underbrace{(x(t) - \tilde{x}(t))}_{e(t)}$$

$L \in \mathbb{R}^{n \times p}$

$\boxed{\dot{e}(t) = (A - LC)e(t)}$ $\longrightarrow$ LINEAR SYSTEM ASYMPTOTICALLY STABLE $\Longleftrightarrow$ E-VALUES $A-LC$ NEGATIVE REAL PART.

$e(0)$ LARGE

$(\lambda_1, \ldots, \lambda_n) \longrightarrow \underbrace{(\lambda - \lambda_1) \ldots (\lambda - \lambda_n)}_{\text{ORDER } n} = \underbrace{\mathrm{DET}[\lambda I - (A - LC)]}_{\text{ORDER } n}$

$\underline{e(t) \longrightarrow 0}$ AS $t \to \infty$

**Theorem**: If the system is observable, then $L$ can be chosen such that eigenvalues of *(A-LC)* have negative real parts.

# Revision of Laplace transforms

05 Continuous LTI systems in frequency domain

# Laplace transform

- Convert real valued functions of real argument to complex valued functions of complex argument

$$f(\cdot): \mathbb{R}_+ \to \mathbb{R} \qquad \xrightarrow{\mathcal{L}} \qquad F(\cdot): \mathbb{C} \to \mathbb{C}$$
$$t \mapsto f(t) \qquad \xleftarrow{\mathcal{L}^{-1}} \qquad s \mapsto F(s)$$

$$\underline{F(s)} = \underline{L\{f(t)\}} = \int_0^\infty f(t)e^{-st}dt$$

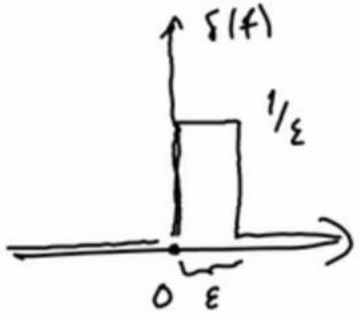
- Assumed *f(t)* such that integral well defined
- Can also be defined for vector/matrix valued functions element by element

$$f(\cdot): \mathbb{R}_+ \to \mathbb{R}^{n\times m} \xrightarrow{\mathcal{L}} F(\cdot): \mathbb{C} \to \mathbb{C}^{n\times m}$$

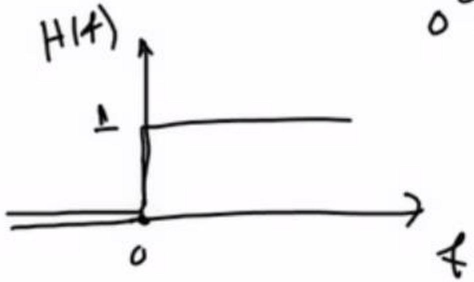
# Laplace transform properties

- Linearity $\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 \mathcal{L}\{f_1(t)\} + a_2 \mathcal{L}\{f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$
- *s*-shift $\mathcal{L}\{f(t) \cdot e^{-\alpha t}\} = F(s+\alpha)$
- Time derivative $\mathcal{L}\{\frac{d}{dt} f(t)\} = sF(s) - f(0)$
- Convolution $\mathcal{L}\{(f * g)(t)\} = F(s) \cdot G(s)$
- Initial value theorem $\lim_{t \to 0} f(t) = \lim_{s \to \infty} s \cdot F(s)$
- Final value theorem $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

ALL LIMITS EXIST.




# Laplace transform of common functions




- Dirac impulse $\mathcal{L}\{\delta(t)\} = 1$
- Step function $\mathcal{L}\{H(t)\} = 1/s$
- Exponential function $\mathcal{L}\{e^{-\alpha t}\} = \frac{1}{s+\alpha}$
- Sinusoidal functions $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$ $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$
- All rational functions
  - Very useful for linear systems (coming up!)
  - Compute inverse Laplace transform by partial fraction expansion

# LTI systems in the frequency domain

05 Continuous LTI systems in frequency domain

# Laplace transform of linear system equations

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m$$

$$y(t) = Cx(t) + Du(t) \qquad y(t) \in \mathbb{R}^p,\ t \in \mathbb{R}$$

$x(0) = x_0$

$$\underbrace{\mathcal{L}\{\dot{x}(t)\}}_{\parallel} = \mathcal{L}\{Ax(t) + Bu(t)\} = A\mathcal{L}\{x(t)\} + B\mathcal{L}\{u(t)\} = AX(s) + BU(s)$$

$$s\mathcal{L}\{x(t)\} - x_0 \Rightarrow \underbrace{(sI - A)}_{(sI-A)^{-1}}X(s) = x_0 + BU(s) \qquad Y(s) = CX(s) + DU(s)$$

$$\boxed{\begin{aligned} X(s) &= (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s) \\ \longrightarrow\quad Y(s) &= C(sI - A)^{-1}x_0 + \left[C(sI - A)^{-1}B + D\right]U(s) \end{aligned}}$$

$$\longrightarrow\ X(s) \in \mathbb{C}^n,\ U(s) \in \mathbb{C}^m,\ Y(s) \in \mathbb{C}^P,\ s \in \mathbb{C}$$

# Comparison to time domain solution

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau$$

$$\underbrace{\mathcal{L}\{x(t)\}}_{X(s)} = \mathcal{L}\{e^{At}\}x_0 + \mathcal{L}\Big\{\underbrace{\int_0^t \underbrace{\underbrace{e^{A(t-\tau)}}_{H(t-\tau)\ \text{IMPULSE TRANSITION}} Bu(\tau)d\tau}_{(H*u)(t)\ \text{CONVOLUTION}}}\Big\}$$

$$\begin{aligned}\mathcal{L}\{(H*u)(t)\} &= \mathcal{L}\{H(t)\}\cdot\mathcal{L}\{u(t)\}\\ &= \mathcal{L}\{e^{At}B\}\cdot U(s)\\ &= \mathcal{L}\{e^{At}\}\cdot B\cdot U(s)\end{aligned}$$

$$\boxed{L\left\{e^{At}\right\} = (sI - A)^{-1}}$$

# Transfer function

$X_0 = 0$

$X(s) = (sI - A)^{-1} B \cdot U(s)$

$Y(s) = C\,X(s) + D\,U(s) = \underbrace{\left[C(sI-A)^{-1}B + D\right]}\,U(s)$

- Zero state response

$$\boxed{G(s) = C\underbrace{(sI - A)^{-1}}B + D}$$

TRANSFER FUNCTION $Y(s) = G(s) \cdot U(s)$

$(sI-A)^{-1} = \dfrac{\mathrm{ADJ}\,(sI-A)}{\underbrace{\mathrm{DET}\,(sI-A)}} \longrightarrow$ ENTRIES = SUBDETERMINANTS OF $(sI-A)$ $(n-1)\times(n-1)$

= POLYNOMIAL OF ORDER AT MOST $n-1$

POLYNOMIAL ORDER $n$
CHARACTERISTIC POL.
OF $A \in \mathbb{R}^{n\times n}$

$(s-p_1) \cdots (s-p_n)$

E-VALUES $A$

$G(s) =$ P×M MATRIX RATIONAL FUNCTIONS OF $s \in \mathbb{C}$

PROPER.

# Transfer function properties

05 Continuous LTI systems in frequency domain

# Transfer function and impulse response

IMPULSE.

- Impulse response $K(t) = Ce^{At}B + D\delta(t)$

$$\mathcal{L}\{K(t)\} = \mathcal{L}\{Ce^{At}B + D\delta(t)\} = C\underbrace{\mathcal{L}\{e^{At}\}}_{(sI-A)^{-1}}B + D\underbrace{\mathcal{L}\{\delta(t)\}}_{1} = C(sI-A)^{-1}B + D = G(s)$$

CONVOLUTION.

IMPULSE RESPONSE

INPUT

- Zero state response

$$y(t) = C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) = (K * u)(t)$$

$$Y(s) = \mathcal{L}\{(K*u)(t)\} = \mathcal{L}\{K(t)\}\cdot\mathcal{L}\{u(t)\} = G(s)\cdot U(s)$$

# Transfer function and stability

$K \leq n$

- Transfer function = proper rational function of $s$

ZEROS

- Single input-single output (SISO system) $G(s) = \dfrac{(s-z_1)(s-z_2)\cdots(s-z_k)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$

POLES ~ CHARACTERISTIC POLY $A \in \mathbb{R}^{n \times n}$

$\Updownarrow$

E-VALUES OF A.

- Denominator = Characteristic polynomial of matrix $A$
- Poles → eigenvalues of matrix $A$ (unless there are pole zero cancellations!)
- Real part of poles determines stability
- Unless there are pole zero cancellations!

# Transfer function and frequency reponse

POLAR COORDS.

- SISO system → Transfer function complex number $G(s) = |G(s)| \cdot e^{\angle G(s)}$
- Apply sinusoidal input → output settles to sinusoid of same frequency

$$u(t) = A\sin(\omega t + \varphi) \longrightarrow \boxed{G(s)} \xrightarrow{y(t)} A \cdot |G(j\omega)| \sin(\omega t + \varphi + \angle G(j\omega))$$

NEW AMPLITUDE

SAME FREQUENCY

NEW PHASE

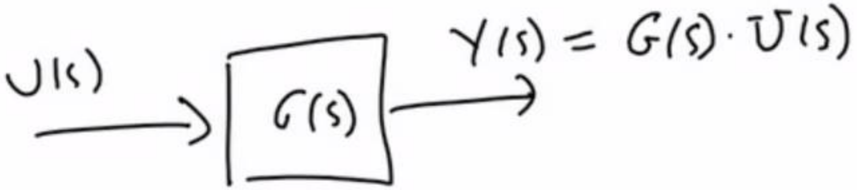
ASYMPTOTICALLY STABLE.

$$U_1(s) + U_2(s) \longrightarrow \boxed{G(s)} \longrightarrow G(s)\,U_1(s) + G(s)\,U_2(s)$$

# Continuous LTI systems in time domain:
# Block Diagrams

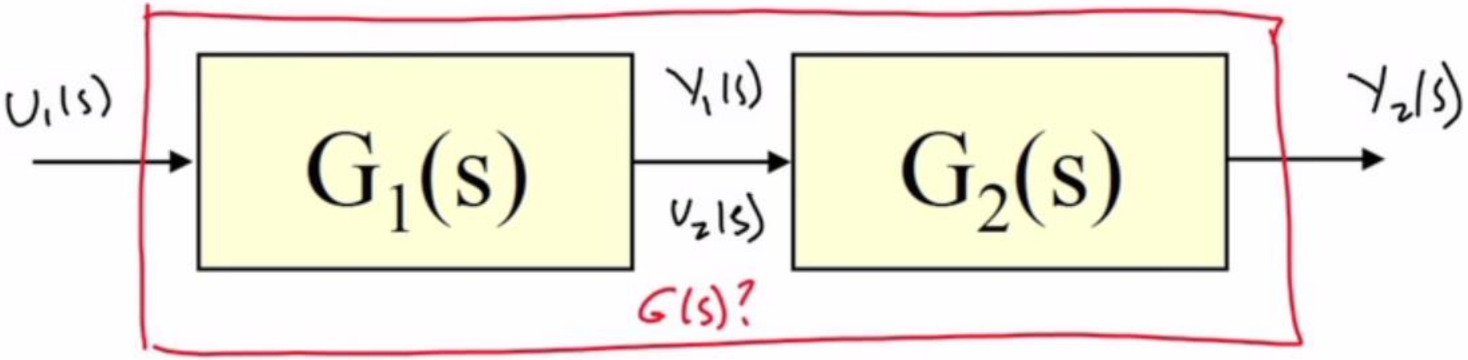
05 Continuous LTI systems in frequency domain

# Cascade interconnection




$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

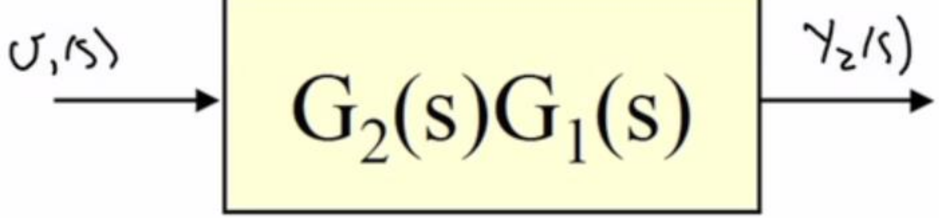
$$\Rightarrow \quad G(s) = C(sI - A)^{-1}B + D$$



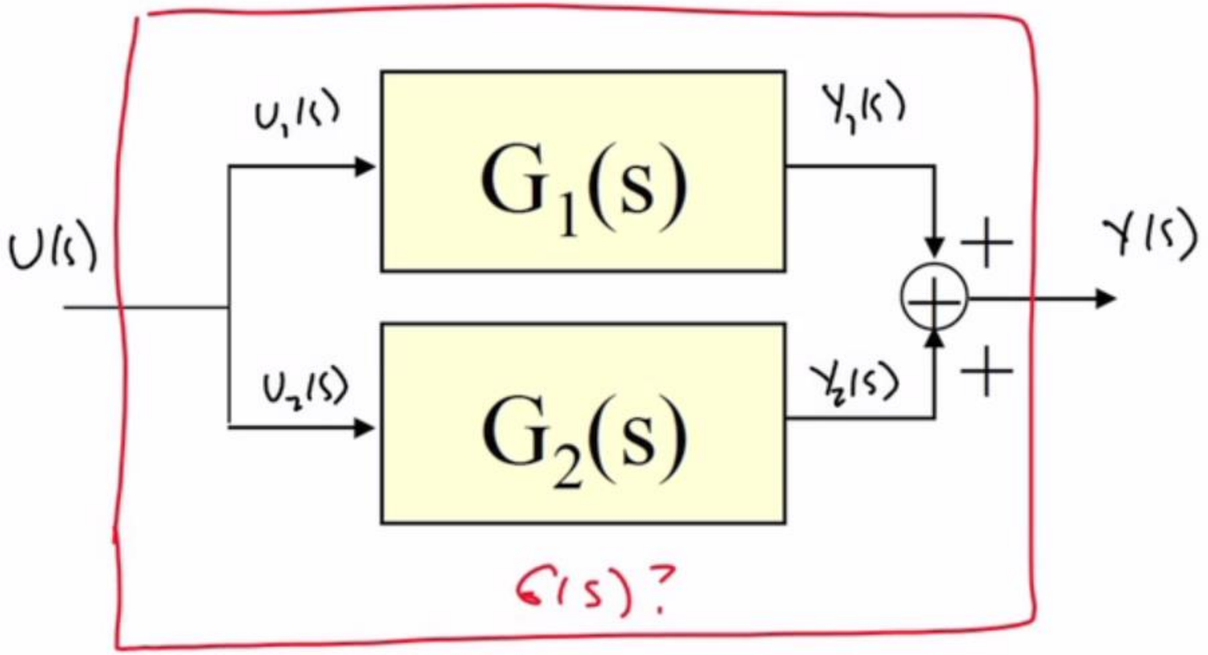

$Y_1(s) = U_2(s)$

$DIM(Y_1(s)) = DIM(U_2(s))$

$$Y_1(s) = G_1(s)\cdot U_1(s) \longleftrightarrow Y_1(s) = U_2(s) \longleftrightarrow Y_2(s) = G_2(s)\cdot U_2(s) = \underbrace{G_2(s)\cdot G_1(s)}_{G(s)}\cdot U_1(s)$$



# Parallel interconnection




$$\mathrm{DIM}[U_1(s)] = \mathrm{DIM}[U_2(s)]$$

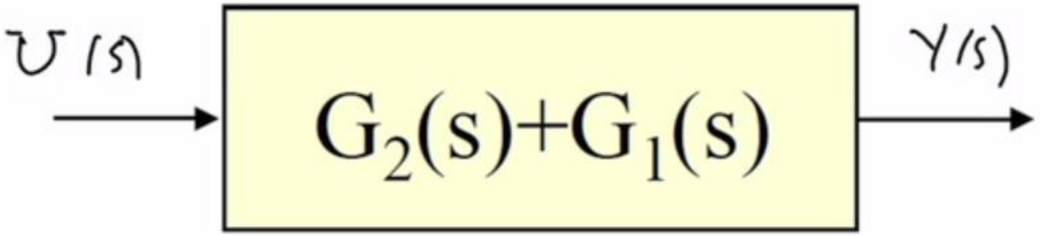
$$\mathrm{DIM}[Y_1(s)] = \mathrm{DIM}[Y_2(s)]$$

$$Y_1(s) = G_1(s) \cdot U_1(s) = G_1(s)\, U(s)$$

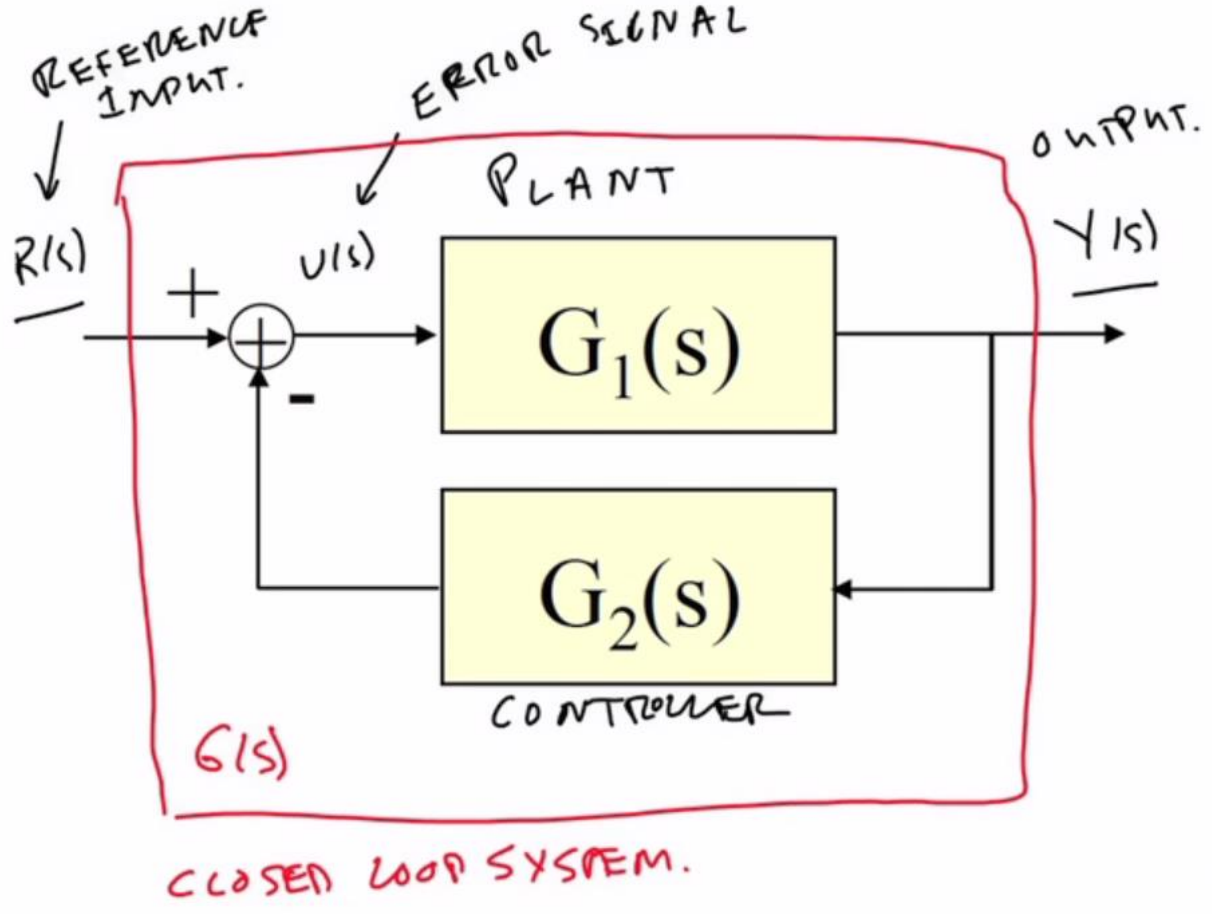
$$Y_2(s) = G_2(s) \cdot U_2(s) = G_2(s)\, U(s)$$

$$Y(s) = Y_1(s) + Y_2(s)$$

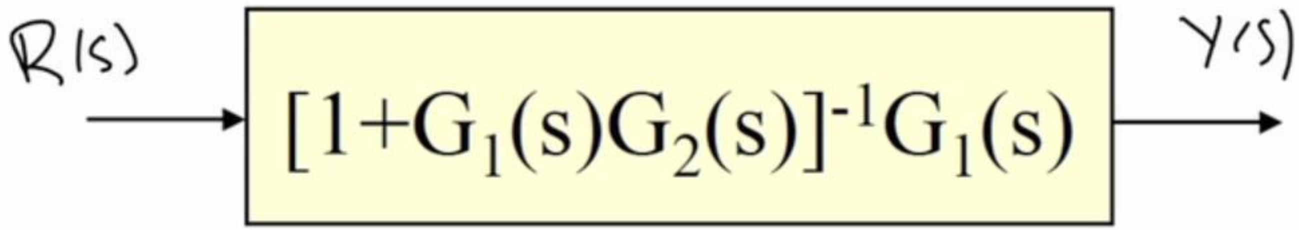
$$Y(s) = \underbrace{\left(G_1(s) + G_2(s)\right)}_{G(s)} U(s)$$



# Feedback interconnection




$$Y(s) = G_1(s)\, U(s)$$
$$= G_1(s) \cdot \left[ R(s) - G_2(s)\, Y(s) \right]$$
$$\left(I + G_1(s) \cdot G_2(s)\right) Y(s) = G_1(s)\, R(s)$$
$$Y(s) = \underbrace{\left[ I + G_1(s)\, G_2(s) \right]^{-1} \cdot G_1(s)}_{G(s)} \cdot R(s)$$



# Sampled Data Systems

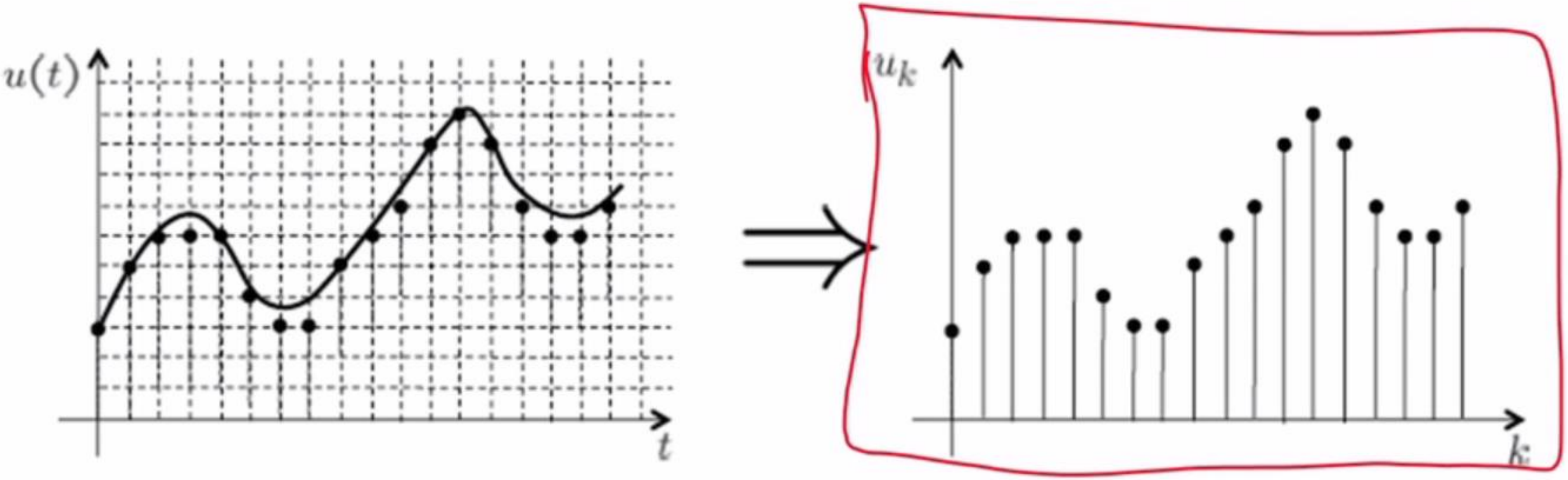
06 Discrete time LTI systems

Embedded computational systems on digital computers

- Measurements of physical quantities measured by computers
- Decisions of computer applied to the physical system

Analog-to-Digital conversion (ADC) and Digital to Analog conversion (DAC)

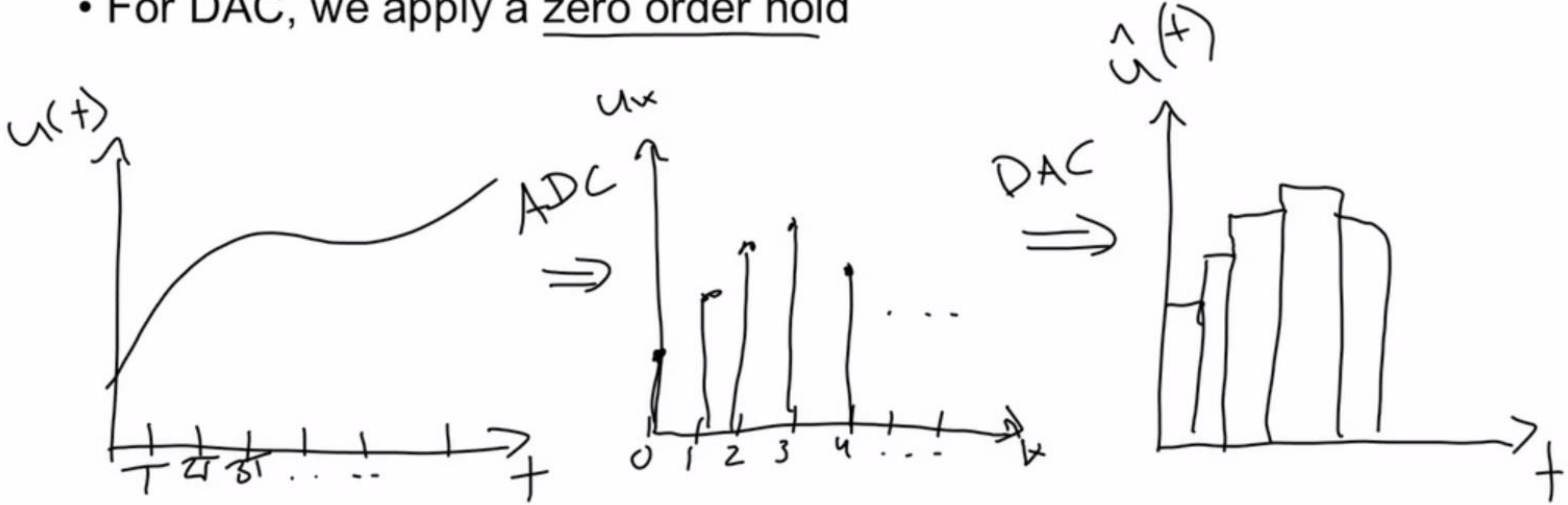
Value and time quantization ESSENTIAL

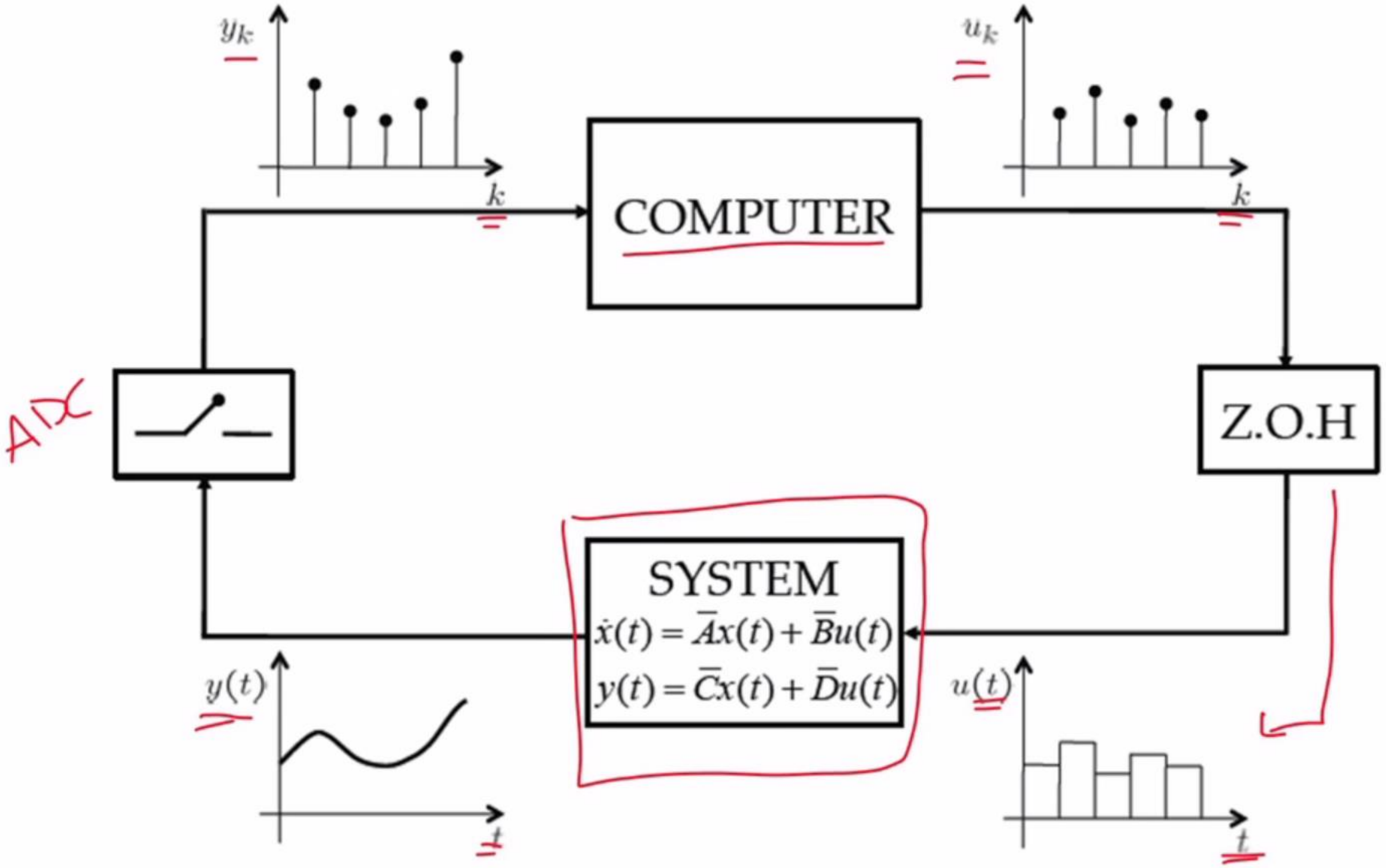


# Value and time quantization

- Often, value quantization is accurate… Focus is then on time quantization.

- Assume
  - For ADC, we sample every T seconds
  - For DAC, we apply a zero order hold





$y_k$
$k$
COMPUTER
$u_k$
$k$
ADC
Z.O.H
SYSTEM
$\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t)$
$y(t) = \bar{C}x(t) + \bar{D}u(t)$
$y(t)$
$t$
$u(t)$
$t$

# Sampled Data <ins>Linear</ins> Systems

06 Discrete time LTI systems

What does a linear system with sampling and zero order hold look like to a digital computer?

$$\boxed{\begin{aligned} \dot{x}(t) &= \bar{A}x(t) + \bar{B}u(t) \qquad & \bar{A} \in \mathbb{R}^{n\times n} \quad \bar{B} \in \mathbb{R}^{n\times m} \\ y(t) &= \bar{C}x(t) + \bar{D}u(t) \qquad & \bar{C} \in \mathbb{R}^{p\times n} \quad \bar{D} \in \mathbb{R}^{p\times m} \end{aligned}}$$

$$u(t) = u_k \quad \text{for all } t \in [kT, (k+1)T)$$

$$y_k = y(kT)$$

Look at solution at at time t.

$$t \in [kT, (k+1)T)$$

$$x(t) = \underbrace{e^{\bar{A}(t-kT)} x(kT)}_{\cancel{ZS}I} + \underbrace{\int_{kT}^{t} e^{\bar{A}(t-\tau)} \bar{B} u(\tau) d\tau}_{ZST}$$

$$x(t) = e^{\bar{A}(t-kT)}x(kT) + \int_{kT}^{t} e^{\bar{A}(t-\tau)}\bar{B}u(\tau)d\tau$$

$$t \to (k+1)T$$

$$x((k+1)T) = e^{\bar{A}T}x(kT) + \left[\int_{kT}^{(k+1)T} e^{\bar{A}((k+1)T-\tau)}\bar{B}d\tau\right]u_k$$

$$= e^{\bar{A}T}x(kT) + \left[\int_{0}^{T} e^{\bar{A}(T-\tau)}\bar{B}d\tau\right]u_k$$

$$y(kT) = \bar{C}x(kT) + \bar{D}u(kT)$$

$$\overset{x_{k+1}}{x\big((k+1)T\big)} = \underset{A}{e^{\bar{A}T}}\,\overset{x_k}{\underline{x(kT)}} + \overbrace{\left(\int_0^T e^{\bar{A}(T-\tau)}\bar{B}\,d\tau\right)}^{B} \underline{u_k}$$

$$\underset{y_k}{y(kT)} = \underset{C}{\bar{C}}\,\underset{x_k}{x(kT)} + \underset{D}{\bar{D}}\,\underset{u_k}{u(kT)}$$

# Discrete Time Linear Systems

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k + Du_k$$

$$x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m, y \in \mathbb{R}^p$$
$$A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$$
$$C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$$

# Discrete Time LTI Systems: A primer

06 Discrete time LTI systems

# Modelling

$$x_{k+1} = Ax_k + Bu_k \qquad x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m, y \in \mathbb{R}^p$$
$$y_k = Cx_k + Du_k \qquad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$$
$$C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$$

What's interesting?

1 – Discrete time variable rather than continuous time variable

2 - The controller is restricted to zero order hold input strategies

# Solution

Given $\hat{x}_0 \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$, $k = 0,1,\ldots,N-1$

$$x_k = \underbrace{\boxed{A^k \hat{x}_0}}_{\text{ZIT}} + \underbrace{\boxed{\sum_{i=0}^{k-1} A^{k-i-1} B u_i}}_{\text{ZST}}$$

What's interesting?

- Both discrete time and continuous time variants have two distinct parts ZIT and ZST

- Hard part is the computation of $A^k$

# Stability

**Theorem 6.1**: System with diagonalizable $A$ matrix is:

- Stable if and only if $\forall i \left|\lambda_i\right| \leq 1$
- Asymptotically stable if and only if $\forall i \left|\lambda_i\right| < 1$
- Unstable if and only if $\exists i : \left|\lambda_i\right| > 1$

Major difference from stability analysis of a continuous time LTI System?

Continuous Time – real part of eigenvalues less than or equal to 0

Discrete Time – absolute value or eigenvalues less than or equal to 1

# Controllability

Consider the controllability matrix P

$$\boxed{P = [B \quad AB \quad A^2B \cdots A^{n-1}B]} \in \mathbb{R}^{\mathrm{n \times nm}}$$

**Theorem 6.4**: The system is controllable if and only if $P$ has rank $n$.

How is this different than a continuous time LTI System?

It's Not!

# Observability

Consider the observability matrix Q

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{\text{np}\times\text{n}}$$

**Theorem 6.5**: The system is observable if and only if $Q$ has rank $n$.

How is this different than a continuous time LTI System?

It's Not!

# Moral of the story?

Analysis tools for continuous time LTI systems and discrete time LTI systems are often similar, and sometimes exactly the same. But mistaking a continuous time system for a discrete time system or vice-versa can be detrimental to a safety critical system, or your exam grade.

# Discrete Time LTI Systems: An example

06 Discrete time LTI systems

# Consider the following discrete time LTI system

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k + Du_k$$

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

Let's address the following:

1. Is the system stable?
2. Is the system controllable?
3. Is the system observable?
4. Compute $A^k$.

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# Stability

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

$$cp = \det(\lambda I - A)$$

$$= \det\left(\begin{bmatrix} \lambda+2 & -1 \\ -3 & \lambda \end{bmatrix}\right)$$

$$= \lambda^2 + 2\lambda - 3$$

$$\underline{0 = (\lambda+3)(\lambda-1)} \Rightarrow \lambda = -3, 1$$

$$|-3| > 1 \Rightarrow \underline{\underline{\text{not stable}}}$$

# Controllability

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det(P) = 0 - 1 = -1 \neq 0$$

$\rightarrow$ P is rank 2

$\rightarrow$ controllable

# Observability

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\det(Q) = 1 - 1 = 0$$

$\rightarrow$ Q rank 1

$\rightarrow$ Not observable

# Compute $A^k$

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

$$A = W \Lambda W^{-1}$$

$$A^k = W \Lambda^k W^{-1}$$

$\lambda = -3, 1$

$$\rightarrow \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$$

$\lambda = -3, \quad Aw = \lambda w$

$$A \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -3w_1 \\ -3w_2 \end{bmatrix} \Rightarrow \begin{matrix} -2w_1 + 3w_2 = -3w_1 \\ 3w_1 = -3w_2 \end{matrix}$$

$$\Rightarrow w = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

# Compute $A^k$

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

$\lambda = 1, \quad Aw = w$

$$-2w_1 + 3w_2 = w_1$$

$$\Rightarrow \quad 3w_1 = w_2$$

$$\longrightarrow \quad w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow W = \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix} \frac{1}{-4} = \begin{bmatrix} -3/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$$

# Compute $A^k$

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

$$A^k = W \Lambda W^{-1}$$

$$= \begin{bmatrix} (-3^k)(3/4) + \frac{1}{4} & (-3^k)(-\frac{1}{4}) + \frac{1}{4} \\ (-3^k)(-3/4) + 3/4 & (-3^k)(\frac{1}{4}) + 3/4 \end{bmatrix}$$

$k = 0 \rightarrow A^0 = I$

$k = 1 \rightarrow A^1 = A$

# Discrete time LTI Systems: Coordinate Change

06 Discrete time LTI systems

# Consider the following discrete time LTI system

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k + Du_k$$

Assume that: Assume $\hat{x}_k = Tx_k$ for some invertible $T \in \mathbb{R}^{n \times n}$

Prove that:

$$\hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}u_k$$
$$y_k = \hat{C}\hat{x}_k + \hat{D}u_k$$

with

$$\boxed{\begin{aligned} \hat{A} &= TAT^{-1}, \quad \hat{B} = TB \\ \hat{C} &= CT^{-1}, \quad \hat{D} = D \end{aligned}}$$

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# Proof

$$T^{-1}T x_k = T^{-1}\hat{x}_k \rightarrow \boxed{x_k = T^{-1}\hat{x}_k}$$

$$x_{k+1} = A x_k + B u_k$$

$$\underbrace{T T^{-1}}_{I} \hat{x}_{k+1} = \underbrace{TAT^{-1}}_{\hat{A}} \hat{x}_k + \underbrace{TB}_{\hat{B}} u_k$$

$$\hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}u_k, \quad \hat{A} = TAT^{-1}, \quad \hat{B} = TB$$

$$y_k = C x_k + D u_k$$

$$y_k = \underbrace{CT^{-1}}_{\hat{C}} \hat{x}_k + D u_k$$

$$y_k = \hat{C}\hat{x}_k + \hat{D}u_k, \quad \hat{C} = CT^{-1}, \quad \hat{D} = D$$

# Nonlinear Systems: Introduction

07 Nonlinear Systems

Linear dynamical systems are modeled by linear differential equations

$$\boxed{\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}}$$

$$x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$$

$$A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$$

Recall that this is a special case of the more general state-space form of dynamical systems modeled in continuous time

$$\boxed{\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned}}$$

$$x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$$

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$$

Here we focus on general nonlinear dynamical systems and in particular

- Autonomous, time invariant systems

$$\boxed{\dot{x}(t) = f(x(t))} \qquad \left(\text{In the linear case } \dot{x}(t) = Ax(t)\right)$$

- Under the assumption that the function f is Lipschitz

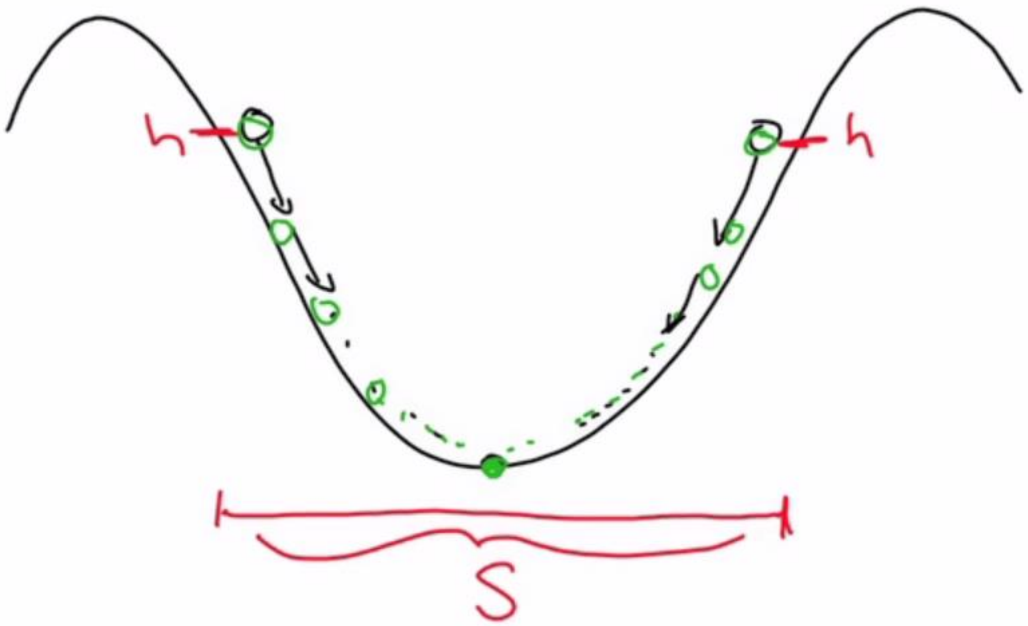
$$\exists \lambda > 0, \forall x, \hat{x} \in \mathbb{R}^n, \quad \left\| f(x) - f(\hat{x}) \right\| \le \lambda \left\| x - \hat{x} \right\|$$

Invariant Sets | Stability

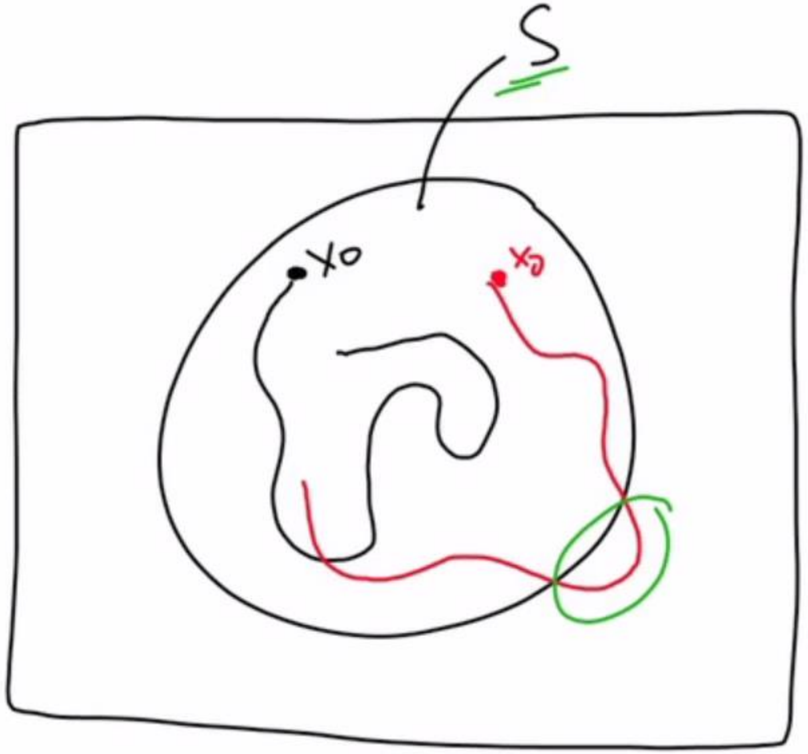
## Invariant Sets (A generalized notion of equilibrium)

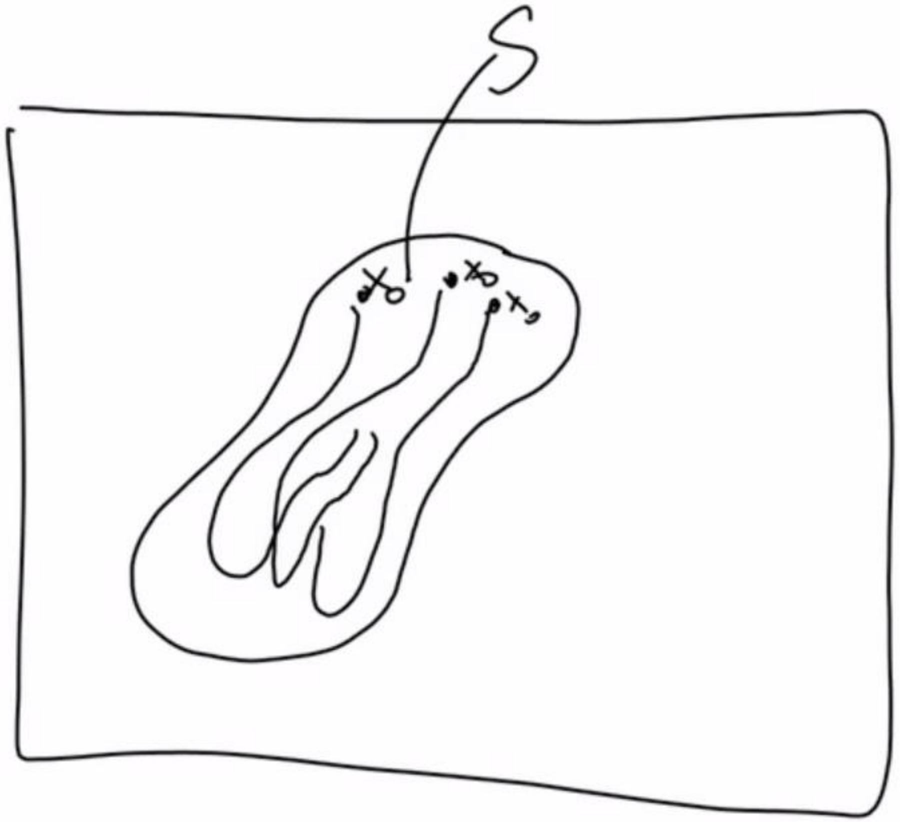
**Definition**: A set of states $S \subseteq \mathbb{R}^n$ is called invariant if

$$\forall x_0 \in S, \forall t \geq 0, \quad x(t) \in S$$



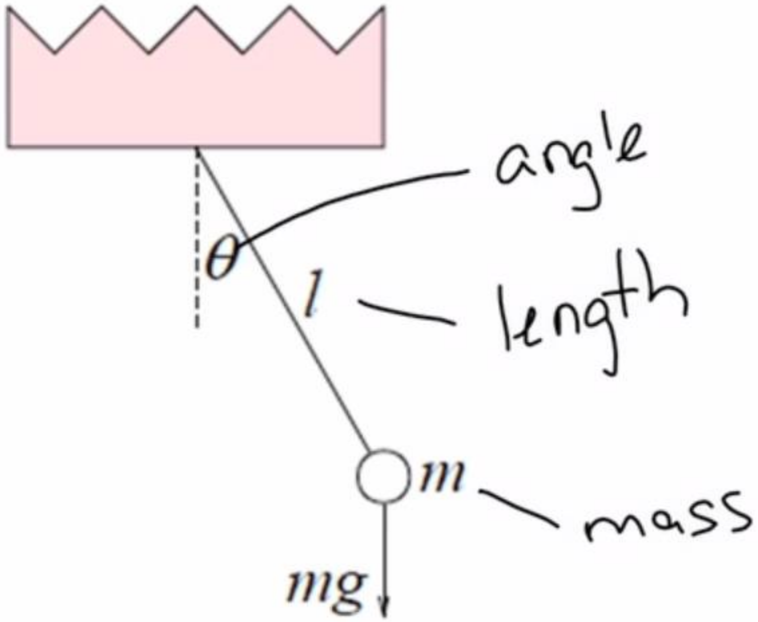
# Invariant Sets (A generalized notion of equilibrium)



# Nonlinear Systems: Modeling a Pendulum

07 Nonlinear Systems

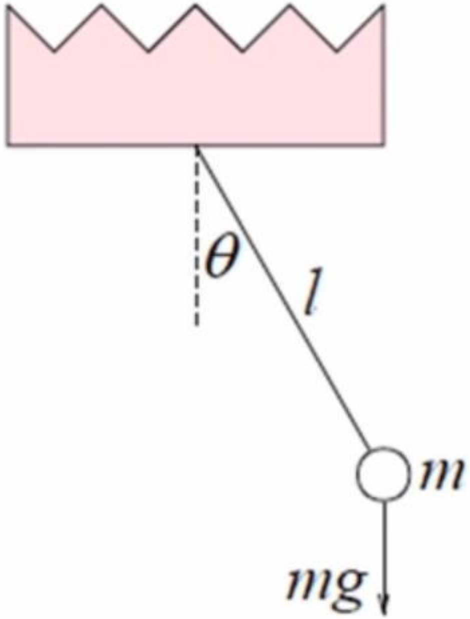


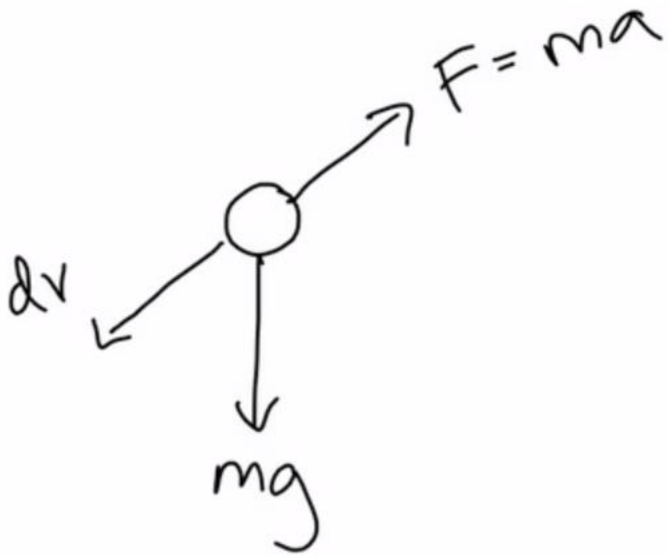

d: friction coefficient

- Derive the equations of motion for a simple pendulum

- Put the equations of motion for a simple pendulum into state space form

## Deriving the equations of motion: Newton's Second Law of Motion




$s = l\theta$

$v = l\dot{\theta}$

$a = l\ddot{\theta}$

$$F = ma = -dv - mg\sin\theta \quad *$$

$$\boxed{ml\ddot{\theta} = -d\dot{\theta} - mg\sin(\theta)}$$

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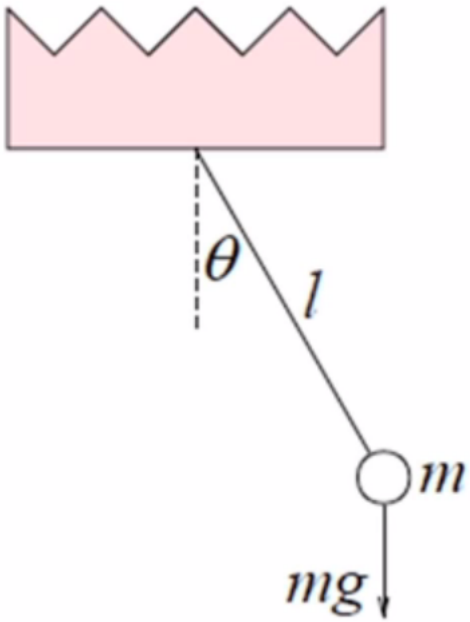


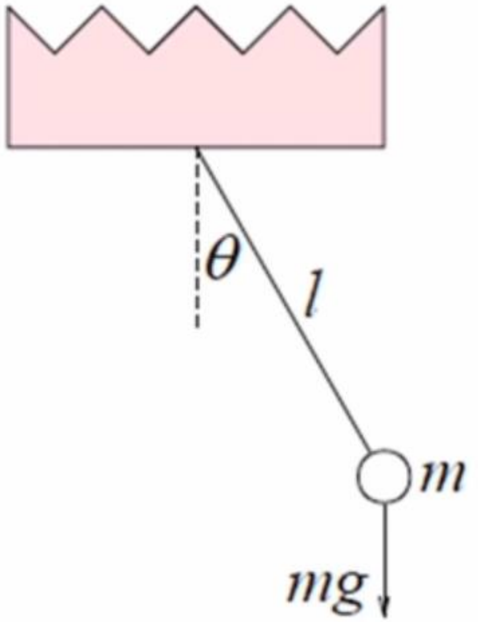

## State Space Representation: Defining the states

$$\dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$x_1(t) = \theta(t)$$

$$x_2(t) = \dot{\theta}(t)$$




## State Space Representation: Final Form

$$\boxed{m l \ddot{\theta}(t) = -d l \dot{\theta}(t) - mg \sin(\theta(t))} \quad *$$

States:
$$\boxed{\begin{aligned} x_1(t) &= \theta(t) \\ x_2(t) &= \dot{\theta}(t) \end{aligned}}$$

$$\dot{x}_1(t) = \dot{\theta}(t) = x_2(t) \implies \dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{\theta}(t) = -\frac{d}{m}\dot{\theta}(t) - \frac{g}{l}\sin(\theta(t))$$

$$= -\frac{d}{m}x_2(t) - \frac{g}{l}\sin(x_1(t))$$

$$\boxed{\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{d}{m}x_2(t) - \frac{g}{l}\sin(x_1(t)) \end{bmatrix}}$$

# Nonlinear Systems: Equilibrium Points

07 Nonlinear Systems

Equilibrium points are invariants sets

**Definition**: A state $\hat{x} \in \mathbb{R}^n$ is called an equilibrium if

$$f(\hat{x}) = 0$$

Linear systems have a linear subspace of equilibria

Sometimes only x=0 is an equilibria

More generally, the null space of the matrix A

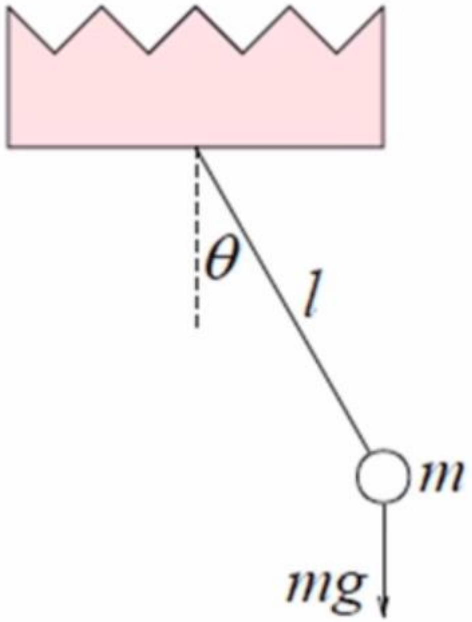
Equilibrium points are invariants sets

**Definition**: A state $\hat{x} \in \mathbb{R}^n$ is called an equilibrium if

$$f(\hat{x}) = 0$$

Nonlinear systems are a bit more complicated

They can have many isolated equilibria




Pendulum dynamical system

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{d}{m}x_2(t) - \frac{g}{l}\sin x_1(t) \end{bmatrix}$$

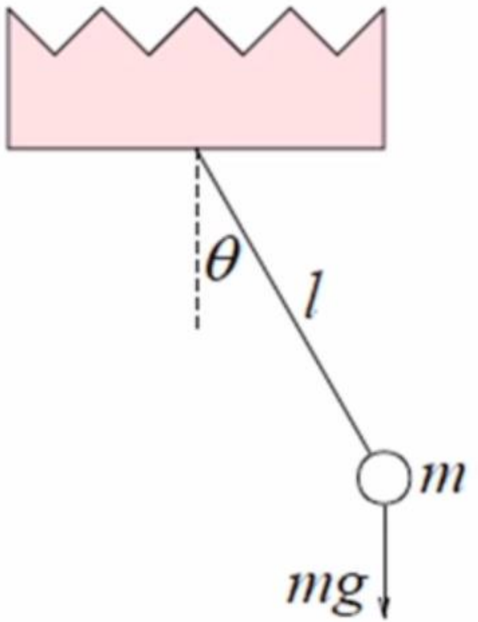
$$\dot{x}_1(t) = 0 \Rightarrow x_2(t) = 0$$

$$\dot{x}_2(t) = 0 = -\cancel{\frac{d}{m}x_2(t)} - \cancel{\frac{g}{l}}\sin(x_1(t))$$

$$\Rightarrow \sin(x_1(t)) = 0$$

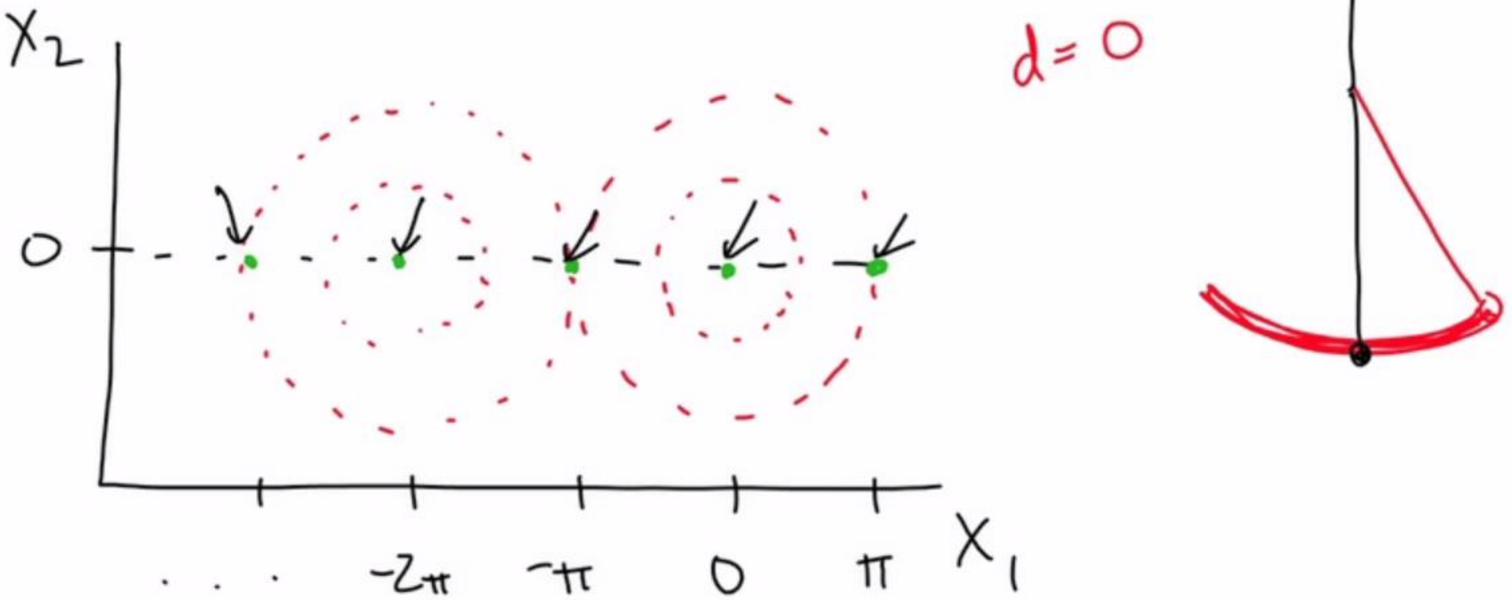
$$\Longrightarrow x_1(t) = k\pi \quad k \in \mathbb{Z}$$

$$\hat{x}_1 = k\pi$$
$$\hat{x}_2 = 0$$
$$k \in \mathbb{Z}$$




Pendulum dynamical system

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{d}{m}x_2(t) - \frac{g}{l}\sin x_1(t) \end{bmatrix}$$



# Nonlinear Systems: Stability

07 Nonlinear Systems

For a nonlinear system

**Definition**: An equilibrium $\hat{x}$ is called stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x_0 - \hat{x}\| < \delta \Rightarrow \|x(t) - \hat{x}\| < \varepsilon \;\; \forall t \geq 0$$

Otherwise equilibrium called unstable.

For a nonlinear system

**Definition**: An equilibrium $\hat{x}$ is called locally asymptotically stable if it is stable and there exists $M>0$ such that

$$\left\|x_0 - \hat{x}\right\| < M \Rightarrow \lim_{t\to\infty} x(t) = \hat{x}$$

It is called globally asymptotically stable if this holds for any $M>0$.
The set of $x_0$ such that $\lim_{t\to\infty} x(t) = \hat{x}$ is called the domain of attraction of $\hat{x}$

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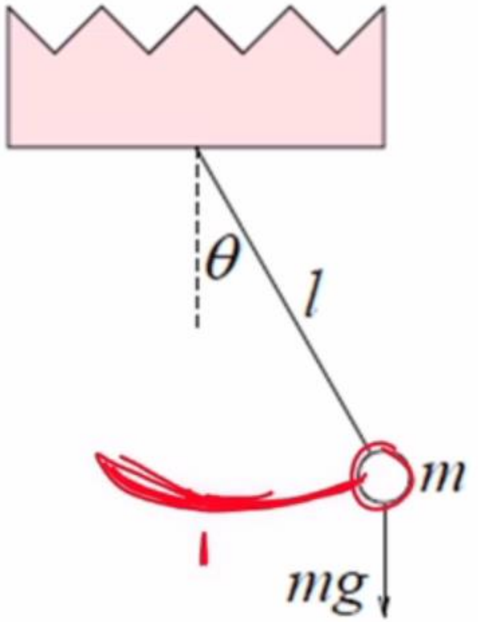
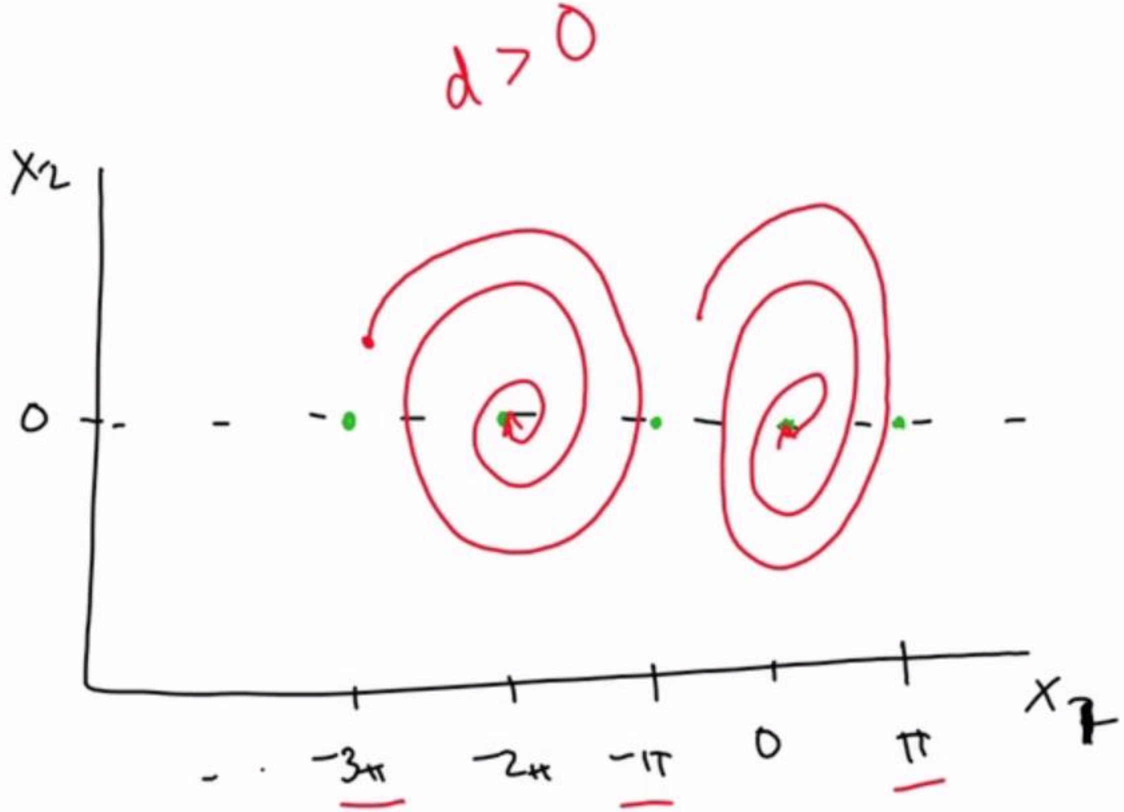


# Linearization

$$\dot{x}(t) = f(x(t)), \quad f(\hat{x}) = 0$$

Approx by linear system

$$f(x) = f(\hat{x}) + A(x - \hat{x}) + \text{higher order terms of } (x - \hat{x})$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\hat{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\hat{x}) \\ \cdot & \cdots & \cdot \end{bmatrix}$$

# Linearization

$$\delta x(t) = x(t) - \hat{x} \in \mathbb{R}^n$$

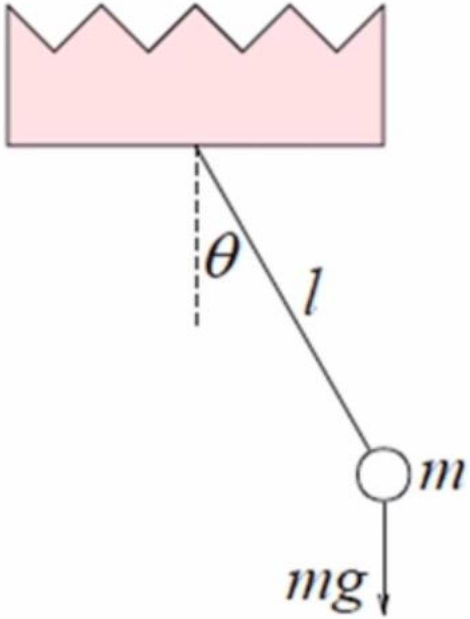
$$\underbrace{\frac{d\,\delta x(t)}{dt} = A\,\delta x(t)}_{\text{good approx. when } \delta x(t) \text{ small}}$$

So, for a good linear approximation

$$\frac{d\delta x(t)}{dt} \approx A\delta x(t)$$

**Theorem 7.1**: The equilibrium $\hat{x}$ is

1. Locally asymptotically stable if the eigenvalues of the linearization have negative real part
2. Unstable if the linearization has at least one eigenvalue with positive real part




Pendulum dynamical system

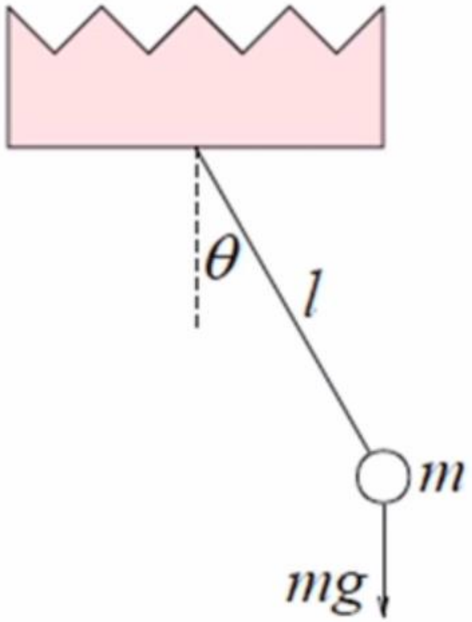
$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{d}{m}x_2(t) - \frac{g}{l}\sin x_1(t) \end{bmatrix}$$

$d > 0$

$$\hat{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{d\,\delta x(t)}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ -g/l & -d/m \end{bmatrix}}_{A} \delta x(t)$$

$$\rightarrow \lambda^2 + \frac{d}{m}\lambda + \frac{g}{l} = 0$$

$\rightarrow$ Neg. real parts $\rightarrow$ Locally A.S.




Pendulum dynamical system

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{d}{m}x_2(t) - \frac{g}{l}\sin x_1(t) \end{bmatrix}$$

$\underline{\underline{d > 0}}$ $\quad \hat{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

$$\frac{d\,\delta x(t)}{dt} = \begin{bmatrix} 0 & 1 \\ g/l & -d/m \end{bmatrix} \delta x(t)$$

$\rightarrow \lambda^2 + \frac{d}{m}\lambda - \frac{g}{l}$

$\rightarrow$ at least $1\ \lambda > 0 \rightarrow$ $\underline{\underline{\text{unstable}}}$