

# Closed-Loop Identification via the Fractional Representation: Experiment Design

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**Abstract**

An important aspect of system identification is the problem of experiment design. This paper uses a fractional representation approach to state and solve the closed-loop experiment design problem in terms of variables which are at the designer's disposal: the closed-loop inputs and the initial controller. Results of computer simulations are presented which compare optimal versus several non-optimal identification experiments.

**Introduction**

The problem of system identification is to estimate the unknown parameters of a dynamical system or plant from measurements of systems input and output as shown in Figure 1a. In the figure,  $u$  is the measurable system input,  $y$  is the measurable system output and  $w$  is an unmeasurable system noise. Experiment design is the problem of choosing the experimental conditions (e.g. input signal  $u$  or sample time) to optimize the results of an identification experiment. In the case of open-loop operation, this is a well studied problem, see [1, 2] and references therein. Unfortunately, most actual identification experiments are conducted while the system is operating under closed-loop control (Figure 1b) and directly applying existing open-loop results to the closed-loop problem generally gives unsatisfactory results. Open-loop techniques can take into account the loop induced  $u-w$  correlation only at the expense of greatly increased complexity and generally yields a result in terms of the plant input  $u$ , a variable which is not at the designer's disposal. In this paper, the fractional representation [3, 4, 5] will be used to avoid the problem of the

plant input and noise correlation and to obtain solutions directly in terms of the loop inputs and the initial controller, variables which are potentially at the designer's disposal.

We assume that during the identification experiment the plant is in stable closed-loop control and that the initial stabilizing controller ( $C_0$  in Figure 1b) is fixed and known. As an experiment design objective we use that proposed in [6]. It assumes that the estimated plant will be used to design a new controller and attempts to minimize errors in the closed-loop dynamics due to the new controller being designed for the estimated plant rather than the true plant. We present the optimal closed-loop input spectra which minimizes this objective subject to power constraints on the loop inputs, the plant input and/or the plant output. To our knowledge, this is the first time the closed-loop experiment design problem has been solved directly in terms of the loop inputs. We also derive the optimal initial controller (for use during the identification experiment) for the case of the a constraint on the power in the plant input and output. This result is a generalization of the results in [7].

**(R, S) Parameterization**

The results presented in this paper are derived by means of the fractional representation [3, 4, 5]. This theory represents both the plant and compensator as the ratio of stable coprime factors and has been used extensively in compensator design, for example [5, 8].

Let  $P_0$  be a noise free plant and  $C_0$  be any initial compensator which stabilizes  $P_0$ . Express  $P_0$  as  $N_0/D_0$  where  $N_0$  and  $D_0$  are stable and coprime (share no unstable zeros) and similarly express  $C_0$  as the ratio of the stable coprime factors  $X_0/Y_0$ . Then all compensators which stabilize  $P$  can be shown [3, 4, 5] to be of the form

$$C_Q = \frac{X_0 + QD_0}{Y_0 - QN_0}, \tag{1}$$

where  $Q$  is stable. Conversely,  $C_Q$ , for any stable  $Q$ , will stabilize  $P$ .

By duality, all noise free plants which are stabilized by a given compensator can be similarly parameterized in terms of a stable parameter, say  $R$ , see [9]. In [10, 11] it is further shown that the noise dynamics of all plants stabilized by a given compensator can also be parameterized, by a stable, stably invertible parameter,  $S$ . These results are restated below.

The standard plant representation for identification is

$$P : y = Gu + Hw. \tag{2}$$

where  $u$ ,  $y$  and  $w$  are the plant input, output and noise respectively.  $G$  and  $H$  are the plant input/output and noise dynamics respectively. Assuming that  $P$  is stabilizable, then, without loss of generality,  $P$  can also be represented as

$$P : Dy = Nu + Mw. \tag{3}$$

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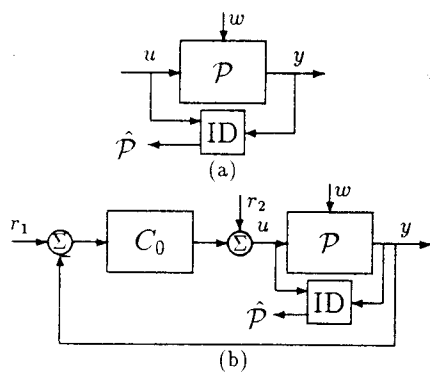


Figure 1: Block diagrams of open-loop and closed-loop system identification problems.

## Experiment Design Problem

where  $N/D$  is a coprime factorization of  $G$ , and  $M$  is both stable and stably invertible. Define the triplet  $(D, N, M)$  to be the coprime factorization of the plant  $\mathcal{P}$ . Then, it can be shown [11] that a given plant  $\mathcal{P}$  with coprime factorization  $(D, N, M)$  is stabilized by the compensator  $C_0$  if and only if it can be expressed as

$$\mathcal{P}_{(R,S)} : y = G_R u + H_{(R,S)} w, \quad (4)$$

where

$$G_R = \frac{N_0 + RY_0}{D_0 - RX_0} = \frac{N}{D}, \quad (5)$$

$$H_{(R,S)} = \frac{S}{D_0 - RX_0} = \frac{M}{D}, \quad (6)$$

$R$  is stable,  $S$  is stable and stably invertible, and  $X_0, Y_0, N_0$ , and  $D_0$  are as defined previously. The parameters  $R$  and  $S$  are easily shown to be

$$R = \frac{D_0 N - N_0 D}{X_0 N + Y_0 D} \quad (7)$$

$$S = \frac{X_0 N_0 + Y_0 D_0}{X_0 N + Y_0 D} M. \quad (8)$$

Figure 2 shows a block diagram representation of  $\mathcal{P}_{(R,S)}$ .

Note that the parameters  $(R, S)$  form a subsystem within the plant:  $\beta = R\alpha + Sw$ . It is the properties of this  $(R, S)$  system which simplifies the closed-loop experiment design problem. These properties include:

1.  $R$  and  $S$  are the only unknowns in the plant/loop. The identification problem can be restated in terms of estimating  $(R, S)$  rather than  $(G, H)$ .
2. The  $(R, S)$  system operates in open-loop. The gain from  $\beta$  to  $\alpha$  is necessarily zero.
3. The input of the  $(R, S)$  system is  $\alpha = X_0 r_1 + Y_0 r_2$  and is thus dependent only on the closed-loop inputs and the compensator. In particular,  $\alpha$  is independent of both the true plant and the plant noise.

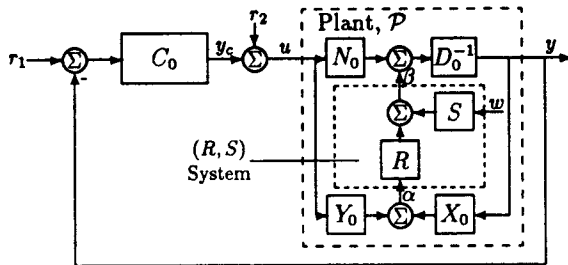


Figure 2: Closed-loop block diagram showing the unknown plant as a bridge centered on the  $(R, S)$  system.

These properties are easily verified using (4), (5), (6). Taken together, these properties allow the identification problem to be restated as one of estimating  $(R, S)$  from  $\alpha$  and  $\beta$  rather than estimating  $(G, H)$  from  $u$  and  $y$ , thus transforming the closed-loop problem in an open-loop problem. In addition, property 3 indicates that  $\alpha$  is dependent only on the closed-loop inputs and the initial compensator, quantities which are at the experiment designer's disposal. Therefore, any experiment design results for the  $(R, S)$  system which gives specifications on  $\alpha$  will also solve the closed-loop experiment design problem.

As an experiment design objective, we use one suggested in [6] which is oriented specifically toward identification for the purposes of controller design. Assuming the estimated plant is used to design a new controller, this objective seeks to minimize errors in the closed-loop dynamics due to the fact that the controller was designed for the estimated plant rather than the true plant.

We make the following assumptions concerning the plant identification experiment.

1. The true plant,  $\mathcal{P}_t$ , is known to be a member of some known set  $\Pi$ . This set is such that there exists a robust controller which stabilizes all plants in  $\Pi$ .
2. During the identification experiment, the plant is controlled by a fixed initial stabilizing controller,  $C_0$ , which stabilizes all plants in  $\Pi$ .  $C_0$  is assumed known.
3. After the identification experiment, the estimated plant,  $\hat{\mathcal{P}}$ , is used to design a new controller by some pre-defined design rule  $\hat{C} = \mathcal{D}(\hat{\mathcal{P}})$ .

The objective to be minimized is defined as

$$J_I = \mathcal{E}_w \int_{-\pi}^{\pi} |\Delta T_{y r_1}|^2 W_T d\omega \quad (9)$$

where  $\mathcal{E}_w$  is the expectation operator with respect to the plant noise,  $W_T$  is a frequency-dependent weighting function, and  $\Delta T_{y r_1}$  is the error in the closed-loop transfer function from  $r_1$  to  $y$  (see Figure 2) as a result of  $\hat{C}$  being designed for  $\hat{\mathcal{P}}$  rather than  $\mathcal{P}_t$ . Thus

$$\Delta T_{y r_1} = \frac{G_t \hat{C}}{1 + G_t \hat{C}} - \frac{\hat{G} \hat{C}}{1 + \hat{G} \hat{C}}. \quad (10)$$

where  $G_t$  and  $\hat{G}$  are the I/O dynamics of the true and estimated plants respectively and

We use a general power constraint on the signal  $\alpha$ :

$$\int_{-\pi}^{\pi} \Phi_{\alpha} W_{\alpha} d\omega \leq K$$

where  $\Phi_{\alpha}$  is the power spectral density of  $\alpha$  and  $W_{\alpha}$  is a frequency-dependent weighting function.

Unfortunately, this objective-constraint pair cannot be solved directly. We therefore approximate the objective following the procedure described by Ljung in [2, 12]. First let  $C_t = \mathcal{D}(\mathcal{P}_t)$  be the controller designed for the true plant. Let  $X_t/Y_t, (D_t, N_t, M_t)$  and  $(\hat{D}, \hat{N}, \hat{M})$  be coprime factorizations of  $C_t, \mathcal{P}_t$  and  $\hat{\mathcal{P}}$  respectively. Finally, let  $(R_t, S_t)$  and  $(\hat{R}, \hat{S})$  be the  $(R, S)$  parameterizations of  $\mathcal{P}_t$  and  $\hat{\mathcal{P}}$ . Assuming that the plant estimate  $\hat{\mathcal{P}}$  is "close" to the true plant  $\mathcal{P}_t$  (in the sense that  $\Delta R = R_t - \hat{R}$  is small) and the design rule  $\mathcal{D}$  is continuous, then it is shown in [11] that  $|\Delta T_{y r_1}|^2$  can be approximated to first order in  $|\Delta R|$  by

$$|\Delta T_{y r_1}|^2 = W_R |\Delta R|^2 \quad (11)$$

where

$$W_R = \left| \frac{X_t Y_t (X_0 N_t + Y_0 D_t)^2}{(X_T N_t + Y_t D_t)^2 (X_0 N_0 + Y_0 D_0)} \right|^2. \quad (12)$$

Thus, the  $J_I$  is approximately

$$J_I \cong \int_{-\pi}^{\pi} \mathcal{E}_w \{ |\Delta R|^2 \} W_R W_T d\omega.$$

Ljung shows in [2] that  $\mathcal{E}_w |\Delta R|^2$  can be approximated asymptotically for large  $n$ , large  $N$  and small  $n/N$  by

$$\mathcal{E}_w |\Delta R|^2 \cong \frac{n |S_t|^2 \Phi_w}{N \Phi_{\alpha}} \quad (13)$$

where  $n$  is the order of the plant model in the identification algorithm,  $N$  is the number of data points used in the identification and  $\Phi_w$  and  $\Phi_\alpha$  are power spectral densities of  $w$  and  $\alpha$  [12]. This approximation assumes that identification algorithm is a PEM (Prediction Error Method) type, that the  $(R_t, S_t)$  system is in open-loop and that the true model is in the model set (i.e. no unmodeled dynamics).

Collecting everything except  $\Phi_\alpha$  into a single objective weighting function,  $W_{obj}$ , the final design problem becomes:

$$\text{Minimize: } J_I = \int_{-\pi}^{\pi} \frac{W_{obj}}{\Phi_\alpha} d\omega \quad (14)$$

with respect to  $\Phi_\alpha$  and possibly  $C_0$  subject to the constraint

$$\int_{-\pi}^{\pi} \Phi_\alpha W_\alpha(\omega) d\omega \leq K \quad (15)$$

where

$$W_{obj} = \frac{n}{N} |S_t|^2 \Phi_w W_R W_T. \quad (16)$$

Solutions to this design problem are discussed in the following section.

## Optimal Designs

We consider two different design problems. In one, the initial controller is assumed fixed and we obtain the set of optimal closed-loop input signals. In the other, we find both the optimal initial controller and the closed-loop input signals. We first consider the constraint in more detail.

### The Constraint

Often the loop inputs represent physical quantities which need to be constrained during the identification experiment. A constraint on  $\alpha$  does not directly yield a constraint on  $r_1$  and  $r_2$  since the inverse map from  $\alpha$  to  $r_1$  and  $r_2$  is not unique. However, if the power apportionment between  $r_1$  and  $r_2$  is determined *a priori* then (15) immediately becomes a constraint on the loop inputs.

For example, suppose that the loop is to be driven from  $r_1$  only. Then,  $\alpha = X_0 r_1$  (with  $r_2 = 0$ ), and a general power constraint on  $r_1$  of the form

$$\int_{-\pi}^{\pi} \Phi_{r_1} W_{r_1}(\omega) d\omega \leq K$$

is accomplished via (15) by using the weighting function

$$W_\alpha = |X_0^{-1}|^2 W_{r_1}. \quad (17)$$

Likewise, if only  $r_2$  is to be used to drive the loop, the appropriate choice of  $W_\alpha$  in (15) is

$$W_\alpha = |Y_0^{-1}|^2 W_{r_2}. \quad (18)$$

Also, a constraint on the total power in the loop inputs is accomplished by setting

$$W_\alpha = |(\max\{X_0(\omega), Y_0(\omega)\})^{-1}|^2. \quad (19)$$

where it has been assumed that each input ( $r_1$  or  $r_2$ ) will be used only at those frequencies where it has the highest gain to the internal signal  $\alpha$ . These weighting functions is listed in Table 1.

It is also common for the loop inputs to be non-physical numbers somewhere inside a feedback control program. In this case constraining the loop inputs has little meaning and it makes more sense to constrain the input and output of the plant itself. Denote that portion of  $u$  and  $y$  which is due only to the loop inputs as  $u_r$  and  $y_r$ . Then, by the proper choice of  $W_\alpha$ , (15) can also be used to constrain the power in  $u_r$  and  $y_r$ . See [11] for details. These weighting function are also listed in Table 1.

Constraint	$W_\alpha$
$\int_{-\pi}^{\pi} \Phi_{r_1} W_{r_1} d\omega \leq K$	$ X_0^{-1} ^2 W_{r_1}$
$\int_{-\pi}^{\pi} \Phi_{r_2} W_{r_2} d\omega \leq K$	$ Y_0^{-1} ^2 W_{r_2}$
Total input power	$ (\max\{X_0, Y_0\})^{-1} ^2$
$\int_{-\pi}^{\pi} \Phi_{y_r} W_{y_r} d\omega \leq K$	$\left  \frac{N_t}{N_t X_0 + D_t Y_0} \right ^2 W_{y_r}$
$\int_{-\pi}^{\pi} \Phi_{u_r} W_{u_r} d\omega \leq K$	$\left  \frac{D_t}{N_t X_0 + D_t Y_0} \right ^2 W_{y_r}$

Table 1: Several possible constraints which can be put in the form (15) and the corresponding  $W_\alpha$ .

### Loop Input Design

The problem of minimizing (14) with respect to  $\Phi_\alpha$  subject to (15) is a standard minimization problem which admits a closed form solution [2]. The solution is

$$\Phi_{\alpha, opt} = \gamma \sqrt{\frac{W_{obj}}{W_\alpha}}. \quad (20)$$

where  $\gamma$  is a scaling constant chosen so that equality is obtained in the constraint. The optimal spectra for the closed-loop inputs  $r_1$  and  $r_2$  must therefore be such that

$$\Phi_{\alpha, opt} = |X_0|^2 \Phi_{r_1, opt} + |Y_0|^2 \Phi_{r_2, opt}. \quad (21)$$

Thus, the optimal closed-loop inputs are characterized not by a single pair of spectra for  $r_1$  and  $r_2$ , but rather by the set of non-negative definite solutions to (21). This solution set can be shown [11] to be independent of the particular choice of the nominal plant  $\mathcal{P}_0$ , and coprime factorizations for  $\mathcal{P}_0$  ( $N_0/D_0$ ), for  $\mathcal{P}_t$  ( $D_t, N_t, M_t$ ), for  $C_t$  ( $X_t/Y_t$ ) and for  $C_0$  ( $X_0/Y_0$ ).

The solution given by (20) and (21) has the disadvantage that it depends on the true plant through  $W_{obj}$  and, in some cases,  $W_\alpha$ . Though this problem is shared by most other experiment design solutions in the literature, it severely limits the solution's practical utility. To address this, we propose to restate the original design problem in terms of an average over all possible plants. The plant has been assumed a member of a known set  $\Pi$ . We further assume that the true plant is a random variable with a known distribution within that set. The averaged design problem can then be stated as:

$$\text{Minimize } J_{I, ave} = \mathcal{E}_{\mathcal{P}_t} \int_{-\pi}^{\pi} \frac{W_{obj}}{\Phi_\alpha} d\omega \quad (22)$$

subject to the average constraint

$$\mathcal{E}_{\mathcal{P}_t} \int_{-\pi}^{\pi} \Phi_\alpha W_\alpha d\omega \leq K, \quad (23)$$

where  $\mathcal{E}_{\mathcal{P}_t}$  is the expectation operator over all possible true plants. Both expectations are assumed finite. Therefore the expectation and integration operator can be exchanged and the solution is easily shown to be

$$\Phi_{\alpha, opt-ave} = \gamma \sqrt{\frac{\mathcal{E}_{\mathcal{P}_t} W_{obj}}{\mathcal{E}_{\mathcal{P}_t} W_\alpha}}. \quad (24)$$

In the rest of the paper, this spectrum will be referred to as the "optimal averaged spectrum". Note that this is not simply the result of averaging the optimal spectrum over  $\Pi$ .

## Controller Design

The previous section considered the problem of finding the optimal closed-loop input signals given a fixed initial controller. In this section we will consider the problem of finding both the optimal closed input signal and the optimal initial controller. The special case considered is that of the design constraint being of "LQG-type" on the power of the plant input and output. In particular we will

$$\text{Minimize } J_I = \int_{-\pi}^{\pi} \frac{W_{obj}}{\Phi_{\alpha}} d\omega \quad (25)$$

with respect to both  $C_0$  and  $\Phi_{\alpha}$ , subject to the constraint

$$\int_{-\pi}^{\pi} (a^2 \Phi_u + b^2 \Phi_y) d\omega \leq K. \quad (26)$$

First note that since the loop is linear, the constraint can be restated as

$$\int_{-\pi}^{\pi} (a^2 \Phi_{u_r} + b^2 \Phi_{y_r}) d\omega \leq K - \int_{-\pi}^{\pi} (a^2 \Phi_{u_w} + b^2 \Phi_{y_w}) d\omega = K_r. \quad (27)$$

where  $K_r$  is defined as shown. As in before,  $u_r$  is that component of  $u$  due entirely to the closed loop inputs and  $u_w$  is that component of  $u$  due to the plant noise  $w$ , likewise with  $y_r$  and  $y_w$ . From Table 1, this can be expressed in the form of (15) with

$$W_{\alpha} = \frac{a^2 |D_t|^2 + b^2 |N_t|^2}{|X_0 N_t + Y_0 D_t|^2}. \quad (28)$$

From the previous section, the optimal  $\Phi_{\alpha}$  for a given  $C_0$  is

$$\Phi_{\alpha, opt|C_0} = \gamma \sqrt{\frac{W_{obj}}{W_{\alpha}}} \quad (29)$$

where  $\gamma$  is chosen such that the equality is achieved in the constraint. Thus  $\gamma$  is

$$\gamma = \frac{K_r}{\int_{-\pi}^{\pi} \sqrt{W_{obj} W_{\alpha}} d\omega}. \quad (30)$$

which implies that the minimum  $J_I$  for any given  $C_0$  is

$$J_{I, min|C_0} = \frac{\left( \int_{-\pi}^{\pi} \sqrt{W_{obj} W_{\alpha}} d\omega \right)^2}{K_r}. \quad (31)$$

Minimizing this with respect to  $C_0$  will yield the optimal initial compensator for the identification experiment. Using (16), (12) and (28),  $W_{obj} W_{\alpha}$  becomes

$$W_{obj} W_{\alpha} = \frac{n}{N} \Phi_w W_T (a^2 |D_t|^2 + b^2 |N_t|^2) \left| \frac{M_t X_t Y_t}{(X_t N_t + Y_t D_t)^2} \right|^2.$$

The important point is that this product is independent of the initial compensator  $C_0$ . Thus, the only term in the cost  $J_{I, min|C_0}$  which is a function of  $C_0$  is  $K_r$ , and  $J_{I, min|C_0}$  will achieve a minimum when  $K_r$  achieves its maximum.  $K_r$  is maximized when

$$J_{LQG} = \int_{-\pi}^{\pi} (a^2 \Phi_{u_w} + b^2 \Phi_{y_w}) d\omega \quad (32)$$

is minimized (see (27)) which is the precisely the LQG problem. Therefore, the optimal controller to use during the identification experiment when the design constraint is of "LQG-type" is the associated LQG controller. The optimal loop input spectrum is given by (29). Note that this result is independent of the weighting function in the objective,  $W_T$ , or the new-compensator design rule  $\mathcal{D}$ .

This result contains, as special cases, two previously published results by Ljung and others [2, 6, 7] on optimal controllers for closed-loop identification and shows them to be the extreme in a continuum of such design problems. In [2, 6] the authors show that if the constraint is placed on  $u$  alone then open-loop operation is optimal for plant identification, assuming that the plant is stable. By a similar analysis it is shown in [7] that a plant output constraint implies a minimum variance controller is optimal, assuming that the plant is minimum phase. Both of these results are special cases of the solution presented here. If  $a = 1$  and  $b = 0$ , the constraint is on  $u$  and minimizing  $J_{LQG}$  yields the "high cost of control" solution (open-loop operation when  $\mathcal{P}$  is stable). If  $a = 0$  and  $b = 1$ , the constraint is on  $y$  and the optimal  $C_0$  is the "cheap control" solution, (the minimum variance controller when  $\mathcal{P}$  is minimum phase). However, the solution presented here is valid if the plant is unstable or non-minimum phase and for the continuum of values of  $a$  and  $b$  between these two extremes.

The original constraint (26) specifies a maximum on the total power of  $u$  and  $y$ . The LQG controller, merely seeks to minimize the noise contribution toward this maximum leaving as much power as possible for the "signal" contribution of  $\alpha$ . Thus this solution can be interpreted as using that  $C_0$  which maximizes the signal-to-noise ratio in the signals  $u$  and  $y$ . This solution also has the implication that the initial controller is important only in the case where the constraint is small enough that the plant noise makes a significant contribution to  $K$ .

## Simulations

To illustrate this experiment design technique we offer the following simulation examples. In these simulations, the plant is stabilized by an initial fixed controller and is identified from 500 input-output data samples. The final estimated plant is used to design a new compensator. The resulting closed-loop transfer functions are calculated for the loops containing the new compensator and the true plant  $T_p$ , and the new compensator and the estimated plant,  $T_{\hat{p}}$ . Finally, the simulated experimental objective is obtained by averaging the integral squared difference between  $T_{\hat{p}}$  and  $T_p$  over 100 independent simulations. Note that this is the original objective (9), not the approximation.

The set of plants we consider was first suggested in [13] as a benchmark problem for system identification, adaptive control and robust control. The continuous time model is

$$y = .9 \left( \frac{1}{s^2} - \frac{1}{s^2 + 2w_r \xi s + w_r^2} \right) u \quad (33)$$

The unknown parameters in the system are  $w_r$ , the natural frequency, and  $\xi$ , the damping ratio, and are assumed to be in the intervals,

$$\begin{aligned} 1 \text{ Hz} &\leq w_r/2\pi \leq 2 \text{ Hz} \\ .02 &\leq \xi \leq .1 \end{aligned} \quad (34)$$

These uncertainties comprise the set  $\Pi$ . When necessary we assume  $w_r$  and  $\xi$  are uniformly distributed. We simulated two particular plants in this set;  $\mathcal{P}_1$  with  $(w_r, \xi) = (1\text{Hz}, 0.1)$  and  $\mathcal{P}_2$  with  $(w_r, \xi) = (2\text{Hz}, 0.02)$ .

This plant was transformed into a discrete plant assuming a zero-order-hold and a sampling rate of 20 Hz. No noise model was included [13]. We assume the full plant dynamics (input/output and noise) are of the form:

$$\mathcal{P} : (1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4}) y = (b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4}) u + w \quad (35)$$

where  $w$  is white Gaussian noise. All eight coefficients are estimated. The initial controller is

$$C_0 = \frac{1 - 2.69z^{-1} + 2.53z^{-2} - .841z^{-3}}{1 - .725z^{-1} - .175z^{-2} - .00925z^{-3}} \quad (36)$$

which stabilizes all plants within the uncertainty (34). With this controller the closed-loop has essentially the same resonance as the plant. This resonance will have a noticeable impact on the final experiment designs.

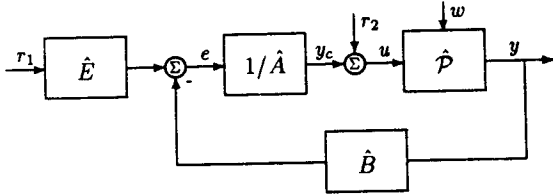


Figure 3: Final control system in STR configuration

The estimated plant is used to design a new controller in the configuration shown in Figure 3. Letting  $\hat{\omega}_r$  be the estimated plant's natural frequency,  $\hat{A}$  and  $\hat{B}$  are chosen so that:

- the resonant poles are radially projected inwards by a factor of .95 (the "control" poles) and .85 (the "estimator" poles), and
- the double integrator poles are moved to a natural frequency of  $(.3 + .3\hat{\omega}_r)$  with a damping ratio of .85 ("control" poles) and a natural frequency of  $(1 + \hat{\omega}_r)$  with a damping ratio of .9 ("estimator" poles).

$\hat{E}$  is then selected so that the four selectable closed-loop zeros coincide with the "estimator" poles specified above. This choice makes the overall control design equivalent to a state-estimator/state-variable-feedback controller. This particular dependence on  $\hat{\omega}_r$  was chosen to obtain a faster closed-loop with the faster plants (those with higher resonant frequencies).

Experiments are designed to minimize

$$J_I = \int_{-\pi}^{\pi} |\Delta T_{vr1}|^2 d\omega.$$

This new controller is slightly more general than that considered earlier due to the presence of the pre-compensator. By similar arguments this objective can be approximated by

$$J_I \cong \int_{-\pi}^{\pi} \frac{W_{obj}}{\Phi_a} d\omega,$$

where  $W_{obj}$  is

$$W_{obj} = \frac{n}{N} \lambda |S_t|^2 \left| \frac{B_t A_t (X_0 N_t + Y_0 D_t)^2}{(B_t N_t + A_t D_t)^2 (X_0 N_0 + Y_0 D_0)} \right|^2 \left| \frac{E_t}{B_t} \right|^2 \quad (37)$$

and  $A_t$ ,  $B_t$ , and  $E_t$  comprise the new compensator designed for the true plant. All other notation is consistent with that used in Section 2 with  $B_t$  and  $A_t$  corresponding to  $X_t$  and  $Y_t$  respectively.

The first set of results we present are for the case where a power constraint is placed on the loop input  $r_2$ . Note that this constraint is one which can not be addressed by previously published experiment design techniques. Figure 4 shows the four input spectra used in the simulations. It includes the optimal

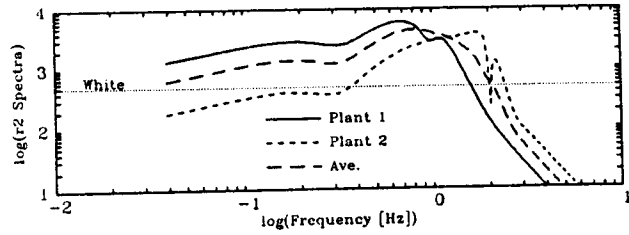


Figure 4: Optimal  $r_2$  spectra assuming the true plant to be  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , the averaged optimal spectrum and a white spectrum. Optimal spectra calculated for a power constraint on  $r_2$ .

spectra assuming the true plant to be  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , the optimal averaged spectrum and a white spectrum, all of unity power. Note how the optimal spectrum for  $\mathcal{P}_2$  notches out the plant's resonant frequency. Recall that when  $C_0$  is used, the plant and loop have the same resonance. Therefore, if  $\mathcal{P}_2$  is the true plant, the gain from  $r_2$  to both  $u$  and  $y$  will be large at 2 Hz; little power is required at that frequency to excite the plant, thus the notch.

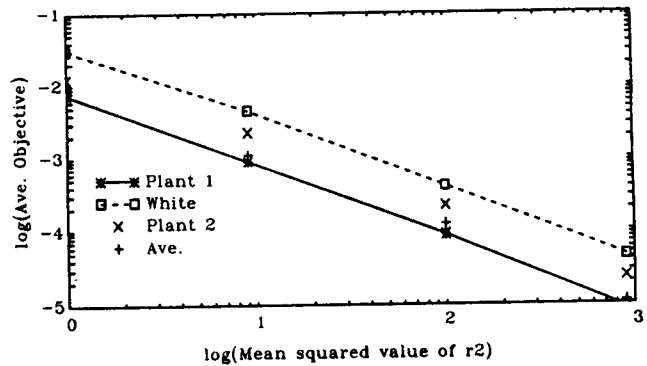


Figure 5: Simulation results obtained when true plant is  $\mathcal{P}_1$  and loop is driven by spectra shown in Figure 4.

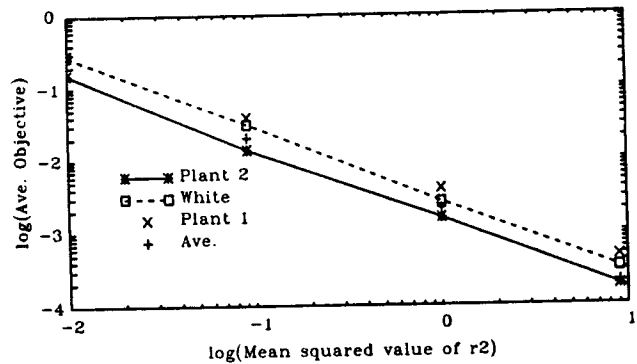


Figure 6: Simulation results obtained when true plant is  $\mathcal{P}_2$  and loop is driven by spectra shown in Figure 4.

Two parallel sets of simulators were run, one with the true plant being  $\mathcal{P}_1$  and the other with  $\mathcal{P}_2$ . Figures 5 and 6 show the average experimental objective versus input power at  $r_2$  for each of the four spectra in Figure 4. In each simulation the plant was also driven by a white noise at  $w$  of power  $\lambda = 10^{-4}$ .

In both sets of simulations, the optimal spectra for the true plant provides the best plant identification, a factor of 3 better than a white input is  $\mathcal{P}_1$  is the true plant and a factor 2 better if  $\mathcal{P}_2$  is the true plant. Note also that if  $\mathcal{P}_2$  is the true plant, using a

spectrum designed to  $\mathcal{P}_1$  is worse than simply a white loop input. However, the optimal averaged input provides a superior plant identification than either a white input or an input designed for the "wrong" plant and nearly as good an identification as the optimal input spectrum.

In a final set of simulations we tested the optimal initial controller result presented earlier. In this example we choose the constraint to be

$$\int_{-\pi}^{\pi} (.5\Phi_u + .5\Phi_y)d\omega \leq K$$

for some  $K$ . The optimal initial controller is the LQG controller designed to minimize the control cost

$$\int_{-\pi}^{\pi} (.5\Phi_{u_w} + .5\Phi_{y_w})d\omega$$

where  $u_w$  and  $y_w$  are those portions of  $u$  and  $y$ , respectively, due to the plant noise alone. To test this result, we performed simulated identification experiments using five different LQG controllers designed to minimize the control cost

$$\int_{-\pi}^{\pi} (\rho\Phi_{u_w} + (1-\rho)\Phi_{y_w})d\omega$$

for  $\rho = 0.99, 0.9, 0.5, 0.1$  and  $0.01$ , each assuming the  $\mathcal{P}_2$  was the true plant. As a sixth alternative, experiments were also simulated using the initial robust controller presented earlier in this section. For the purposes of this example the LQG controllers will be referred to as  $C_{0,\rho}$  and the robust controller will be denoted  $C_{0,rob}$ .

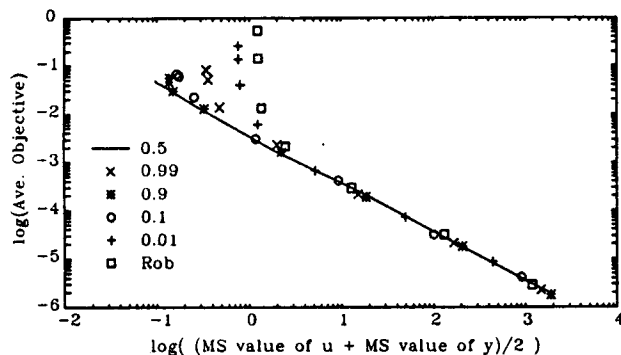


Figure 7: Simulation for the initial controllers  $C_{0,99}$  through  $C_{0,01}$  and  $C_{0,rob}$ . In all simulations the true plant is  $\mathcal{P}_2$ .

Figure 7 shows the results of simulations using each of these six initial controllers. In all cases,  $\mathcal{P}_2$  was the true plant. As before, each data point is the average of 100 independent simulations. These simulations confirm that  $C_{0,5}$  is the best initial controller, providing the lowest average objective over 4 orders of magnitude in both the constraint and the objective. Just as importantly, these results confirm that the initial controller is important only if the constraint is sufficiently small that the plant noise is a significant component of the plant input and output.

## Conclusion

The problem of system identification experiment design in which the system to be identified is operating in a stable closed-loop has been addressed. We have shown how to minimize the objective

proposed by Yuan and Ljung [6] subject to power constraints on the closed-loop inputs, the plant input and/or the plant output. In addition to being more general than other solutions, our solution has the unique feature that it yields specifications on variables which are at the designer's disposal, i.e. the closed-loop inputs and the initial compensator used during the experiment. Previous solutions yield specifications on the plant input, a variable which not at the designer's disposal. The solution for the optimal initial compensator contains two previously published results as special cases and shows them to be the extremes of a continuum of such solutions. We also presented simulation results which demonstrate the use of our method.

In this paper, we have not addressed the important problem of unmodeled dynamics. However, we believe that this method offers advantages here as well. For example, the weighting function  $W_T$  can be selected to de-emphasize those frequencies were the unmodeled dynamics are dominate. Furthermore, the fact that the  $(R, S)$  system is necessarily stable simplifies the techniques for on-line uncertainty estimation in [14] and broadens the techniques in [15, 16] to be applicable to any system operating in a stable closed-loop.

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