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AUTOMATIC CONTROL LABORATORY

OPEN HOUSE

Automatic Control Laboratory

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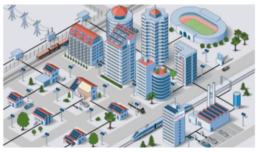
Tuesday, 28. Nov. 2023

17:15, ETF E1

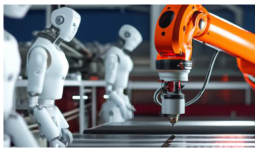


Control of energy systems and smart grids

Control: a world of data

Reinforcement learning and optimal control

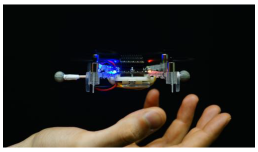
Game-theoretical control of socio-technical systems



Industrial control and robotics

Optimization, games, and learning in closed loop

RoboCup – NomadZ team



Experimental validation of control methods

System Identification

Lecture 9: Prediction error methods & ARX models

Roy Smith

Identification framework

Measurement data

Define the measurement data set:

$$Z_K = \{u(0), y(0), \ldots, u(K-1), y(K-1)\}.$$

Objective

Define the cost function (includes the model structure)

$$J(\theta, Z_K)$$

General optimisation formulation

$$\underset{\theta}{\text{minimise}}\, J(\theta, Z_K) \quad \text{or} \quad \hat{\theta} = \text{argmin}\, J(\theta, Z_K).$$

Optimisation objectives

Residual error objectives

error: $e(k, \theta) = y(k) - G(\theta)u(k).$

$J(\theta) = \|e(\theta)\|_2^2$ or $\|e(\theta)\|_\infty$, or $\|e(\theta)\|_1$?

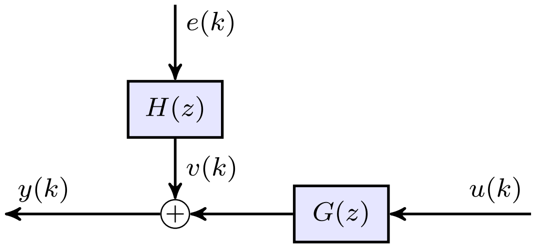
Parametric error objectives

$J(\theta) = \|\theta - \theta_0\|_2$ or $E\{\theta - \theta_0\}$

Prediction error objective

$$J(\theta) = E\{y(k+1) - \hat{y}(k+1, \theta|k)\}$$

Prediction



Typical assumptions

$G(z)$ and $H(z)$ are stable,

$H(z)$ is stably invertible (no zeros outside the unit disk)

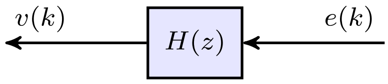
$e(k)$ has known statistics: known pdf or known moments.

One-step ahead prediction

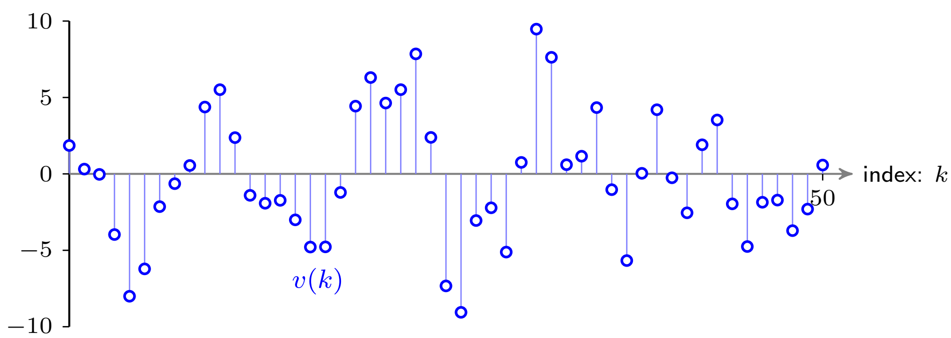
Given $Z_K = \{u(0), y(0), \ldots, u(K-1), y(K-1)\}$,

what is the best estimate of $y(K)$?

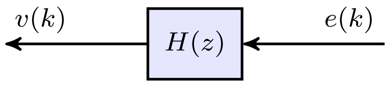
Prediction



What is the next output, $v(k)$?



Prediction



Noise model invertibility

Given, $v(k)$, $k = 0, \ldots, K-1$, can we determine $e(k)$, $k = 0, \ldots, K-1$?

Inverse filter: $H_{\mathsf{inv}}(z)$: $$e(k) = \sum_{i=0}^{\infty} h_{\mathsf{inv}}(i) v(k-i)$$

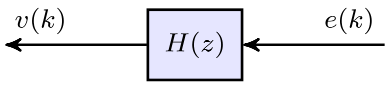
We also want the inverse filter to be causal and stable:

$$h_{\mathsf{inv}}(k) = 0, k < 0, \quad \text{and} \quad \sum_{k=0}^{\infty} |h_{\mathsf{inv}}(k)| < \infty.$$

If $H(z)$ has no zeros for $|z| \geqslant 1$, then,

$$H_{\mathsf{inv}}(z) = \frac{1}{H(z)}.$$

Prediction



One step ahead prediction

Given measurements of $v(k)$, $k = 0, \ldots, K-1$, can we predict $v(K)$?

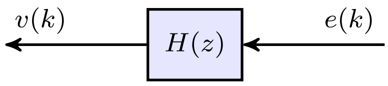
Assume that we know $H(z)$, how much can we say about $v(K)$?

Assume also that $H(z)$ is monic ($h(0) = 1$).

$$v(k) = \sum_{i=0}^{\infty} h(i) e(k-i)$$

$$= e(k) + \underbrace{\sum_{i=1}^{\infty} h(i) e(k-i)}_{\substack{= m(k-1) \\ \text{"observed"}}}$$

Prediction



One-step ahead prediction

The prediction of $v(k)$, based on measurements up to time $k-1$ is,

$$\hat{v}(k|k-1).$$

We will argue that a good choice in this case is,

$$\hat{v}(k|k-1) = m(k-1) = \sum_{i=1}^{\infty} h(i)e(k-i).$$

The error in our prediction is $e(k)$ — which we clearly can't reduce.

One-step ahead prediction statistics

General case

Say $e(k)$ is i.i.d. with probability density function: $f_e(x)$,

A posteriori distribution

What are the statistics of $v(k)$ given $v_{-\infty}^{k-1} = \{v(-\infty), \ldots, v(k-1)\}$?

$$v(k|k-1) \;=\; e(k) \;+\; m(k-1),$$

so $f_{v(k|k-1)}(x) \;=\; f_e(v(k) - m(k-1)).$ $\longleftarrow$ $e(k)$ density shifted by $m(k-1)$.

Choosing an estimate: the expected value

$$\hat{v}(k|k-1) \;=\; E\{v(k|k-1)\} \;=\; m(k-1) + E\{e(k)\} \;=\; m(k-1).$$

One-step ahead prediction

Calculation

$$\hat{v}(k|k-1) = m(k-1) = \sum_{i=1}^{\infty} h(i)e(k-i)$$
$$= (H(z)-1)\,e(k) \qquad \text{(assuming } H(z) \text{ is monic)}$$
$$= \frac{H(z)-1}{H(z)} v(k)$$
$$= (1 - H_{\text{inv}}(z))\,v(k)$$
$$= -\sum_{i=1}^{\infty} h_{\text{inv}}(i)v(k-i)$$

Note that $\hat{v}(k|k-1)$ depends only on values up to time $k-1$.

The best we can do is:

$$\hat{v}(k|k-1) = -\sum_{i=1}^{k} h_{\text{inv}}(i)v(k-i) \approx -\sum_{i=1}^{\infty} h_{\text{inv}}(i)v(k-i).$$

Example

Moving average model

$$v(k) = e(k) + ce(k-1), \quad \Longrightarrow \quad H(z) = 1 + cz^{-1}.$$

For $H(z)$ to be stably invertible we require $|c| < 1$.

$$H_{\text{inv}}(z) = \frac{1}{1+cz^{-1}} = \sum_{i=0}^{\infty}(-c)^i z^{-i}.$$

One-step ahead predictor

$$\hat{v}(k|k-1) = (1 - H_{\text{inv}}(z))v(k) = -\sum_{i=1}^{\infty}(-c)^i v(k-i) \approx -\sum_{i=1}^{k}(-c)^i v(k-i)$$
$$= cv(k-1) - c^2 v(k-2) + c^3 v(k-3) + \cdots - (-c)^k v(0).$$

Exercise: show $\hat{v}(k|k-1) = c \underbrace{(v(k-1) - \hat{v}(k-1|k-2))}_{\epsilon(k-1)}$ (prediction error at $k-1$)

$$= c\epsilon(k-1)$$

Another example

Autoregressive noise model

Our noise model is:

$$v(k) = \sum_{i=0}^{\infty} a^i e(k-i) \qquad |a| < 1 \text{ for stability.}$$

So, $H(z) = \sum_{i=0}^{\infty} a^i z^{-i} = \frac{1}{1 - az^{-1}}$,

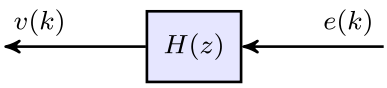
and $H_{\mathsf{inv}}(z) = 1 - az^{-1}$ (a moving average process)

Our one-step ahead predictor is,

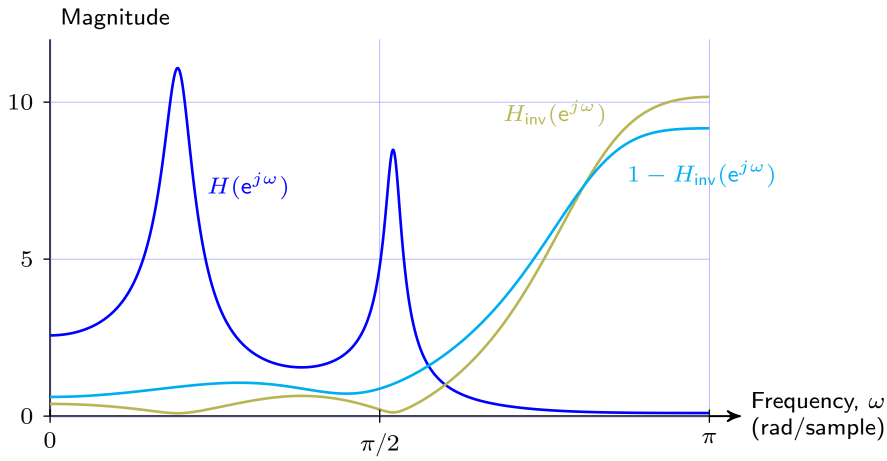
$$\hat{v}(k|k-1) = (1 - H_{\mathsf{inv}}(z))v(k) = av(k-1).$$

A more complicated example

Noise model

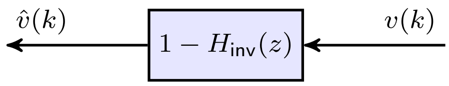


$$H(z) = \frac{z^4 + 0.5z^3 + 0.1z^2 + 0.1z + 0.1}{z^4 - 1.4z^3 + 1.6z^2 - 1.3z + 0.8}$$

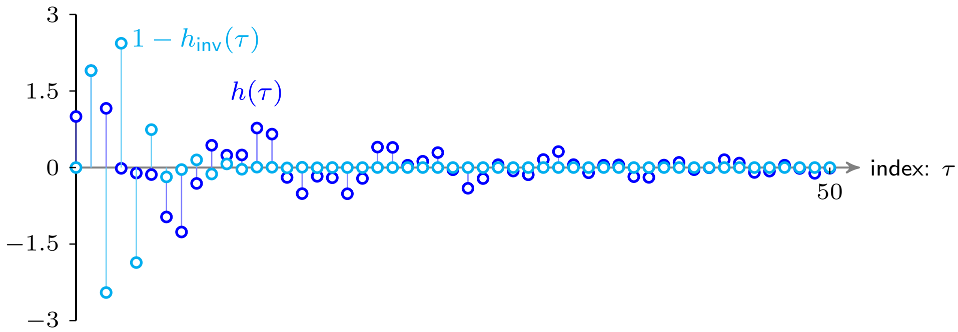


A more complicated example

Predictor

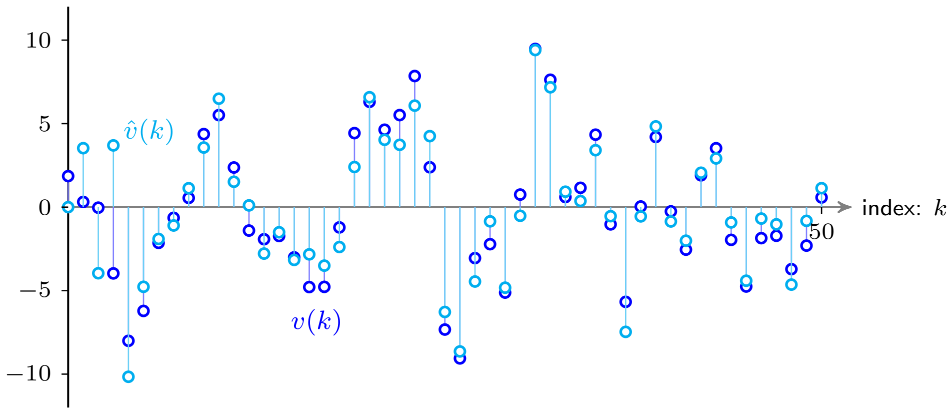


Impulse responses: $H(z)$ and $1 - H_{\mathsf{inv}}(z)$



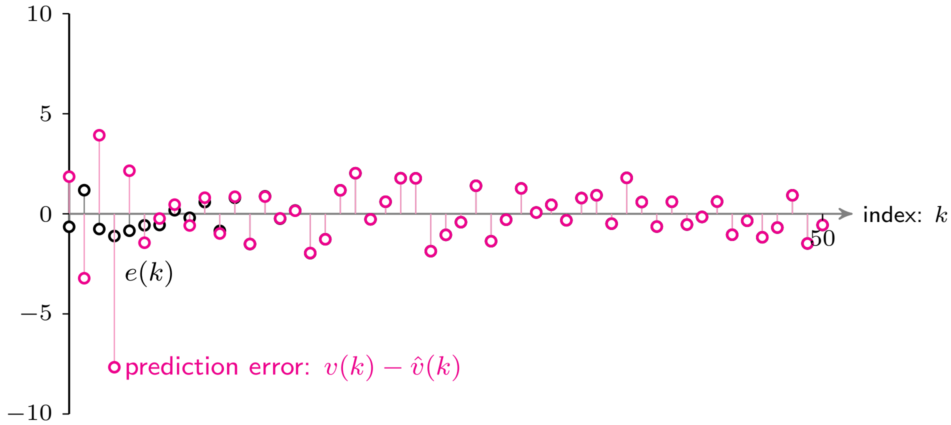
A more complicated example

One-step ahead predictions



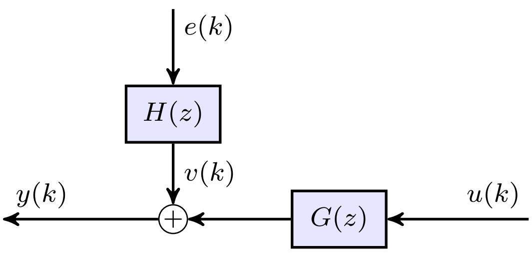
A more complicated example

One-step ahead prediction errors



Output prediction

$$y(k) = G(z)u(k) + v(k)$$



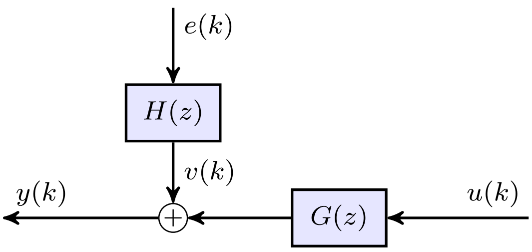
One-step ahead prediction

Choose the expected value of the conditional distribution,

$$\begin{aligned}\hat{y}(k|k-1) &= E\{y(k|k-1)\} = G(z)u(k) + \hat{v}(k|k-1) \\ &= G(z)u(k) + (1 - H_{\text{inv}}(z))v(k) \\ &= H_{\text{inv}}(z)G(z)u(k) + (1 - H_{\text{inv}}(z))y(k)\end{aligned}$$

Output prediction

$$y(k) = G(z)u(k) + v(k)$$



Prediction error

$$\begin{aligned} y(k) - \hat{y}(k|k-1) &= -H_{\mathsf{inv}}(z)G(z)u(k) + H_{\mathsf{inv}}(z)y(k) \\ &= H_{\mathsf{inv}}(z)(y(k) - G(z)u(k)) = H_{\mathsf{inv}}(z)v(k) \\ &= e(k) \end{aligned}$$

The innovation is the part of the output prediction that cannot be estimated from past measurements.

Prediction error based identification

Prediction error

The one-step ahead predictor is parametrised by θ,

$$\hat{y}(k|\theta, Z_K) = H_{\mathsf{inv}}(\theta, z)G(\theta, z)u(k) + (1 - H_{\mathsf{inv}}(\theta, z))y(k)$$

Define a parametrised prediction error,

$$\epsilon(k, \theta) \;=\; y(k) - \hat{y}(k, \theta) \;=\; -H_{\mathsf{inv}}(\theta, z)G(\theta, z)u(k) \;+\; H_{\mathsf{inv}}(\theta, z)y(k).$$

Prediction error-based identification

Typical cost function: $\quad J(\theta, Z_K) \;=\; \|\epsilon(k, \theta)\|_2.$

Pick θ to minimise the error.

Optimisation formulation: $\quad \hat{\theta} \;=\; \underset{\theta}{\operatorname{argmin}} \; J(\theta, Z_K).$

ARX and ARMAX models

Autoregressive moving average models

Model form (without noise):

$$G(z) = \frac{b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}.$$

Input output reationship:

$$\begin{aligned} y(k) &= G(z)u(k) \\ &= -a_1 y(k-1) - \cdots - a_n y(k-n) + b_1 u(k-1) + \cdots + b_m u(k-m) \\ &= \varphi^T(k)\theta \end{aligned}$$

where

$$\varphi(k) = \begin{bmatrix} -y(k-1) & \cdots & -y(k-n) & u(k-1) & \cdots & u(k-m) \end{bmatrix}^T$$

$$\theta = \begin{bmatrix} a_1 & \cdots & a_n & b_1 & \cdots & b_m \end{bmatrix}^T.$$

ARX and ARMAX models

Autoregressive moving average models

$$y(k) = G(\theta, z)u(k) = \varphi^T(k)\theta.$$

where $\varphi(k)$ is called the regressor vector
θ is the parameter vector.

Because we can calculate $y(k)$ from the model, with knowledge of past $y(k)$, $u(k)$ and model parameters, θ, the output $y(k)$ is sometimes written as,

$$y(k) = \varphi^T(k)\theta = y(k|\theta),$$

to emphasize the parameter dependence.

ARX and ARMAX models

Regressor formulation

Least squares estimation

Model framework (ARX or ARMAX)

Consider $y(k) = \varphi^T(k)\theta$ for $k = 0, \ldots, N-1$.

$$\underbrace{\begin{bmatrix} y(0) \\ \vdots \\ y(N-1) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} \varphi^T(0) \\ \vdots \\ \varphi^T(N-1) \end{bmatrix}}_{\Phi} \theta.$$

or, in matrix form, $Y = \Phi\theta$

If $N > (n+m)$ this system is overdetermined.

Least squares solution

$$\hat{\theta} = \left(\Phi^T\Phi\right)^{-1}\Phi^T Y$$ (MATLAB: `Theta = Phi\Y;`)

Least squares estimation

Optimisation framework

The least squares solution, $\hat{\theta} = \left(\Phi^T \Phi\right)^{-1} \Phi^T Y$, solves the problem,

$$\begin{aligned} \underset{\theta}{\text{minimise}} \quad & \|\epsilon\|_2 \\ \text{subject to} \quad & Y = \Phi\theta + \epsilon \end{aligned}$$

Define the cost function: $J(\theta, Z_N) = \|Y - \Phi\theta\|_2 = \|\epsilon\|_2$

General formulation:

$$\begin{aligned} \hat{\theta} &= \underset{\theta}{\operatorname{argmin}}\, J(\theta, Z_N) \\ &= \underset{\theta}{\operatorname{argmin}}\, \|\epsilon\|_2 \\ & \quad \text{subject to}: \quad \epsilon = Y - \Phi\theta. \end{aligned}$$

Example

Single state system: 2 parameters

System: $G(z, \theta) = \dfrac{bz^{-1}}{(1 + az^{-1})} = \dfrac{b}{(z + a)}$ where $\theta = \begin{bmatrix} a \\ b \end{bmatrix}$.

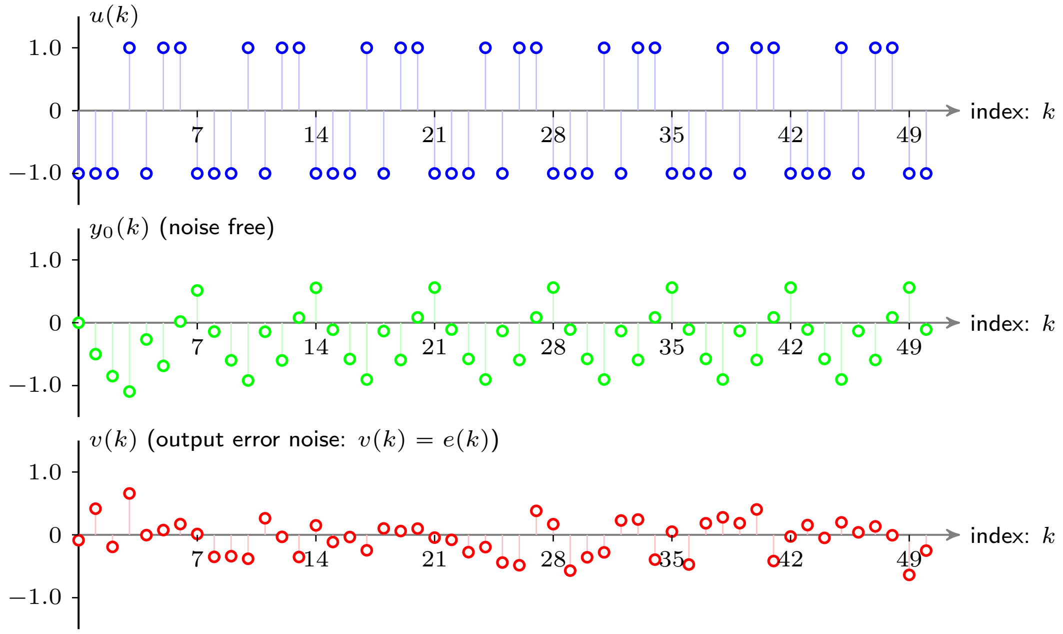
Difference equation: $y(k) = -ay(k-1) + bu(k-1)$.

Least squares solution:

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \left(\Phi^T \Phi\right)^{-1} \Phi^T Y$$

Example

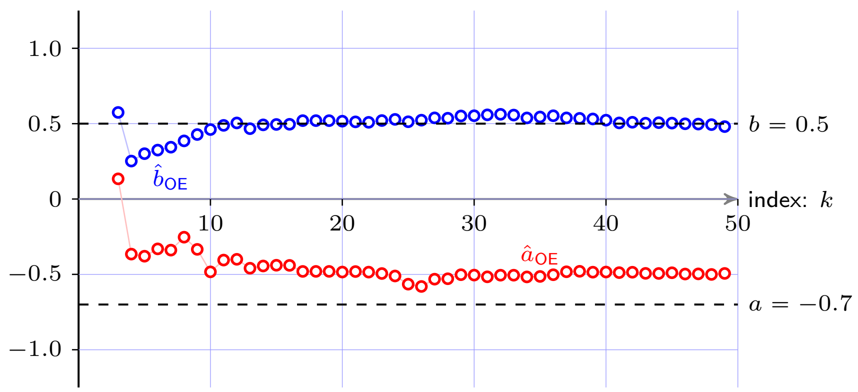
Output error experiment



$\mathrm{var}(e(k)) = 0.25^2$

Example

Least-squares estimates: output error structure



True system

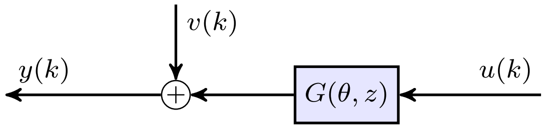
$$G(z) = \frac{bz^{-1}}{1 + az^{-1}}, \quad H(z) = 1.$$

Estimation:

$$y(k) = -\hat{a}_{\mathsf{OE}} y(k-1) + \hat{b}_{\mathsf{OE}} u(k-1)$$

Error statistics and noise statistics

Experimental configuration: output noise



Estimation framework

$$\begin{bmatrix} y(0) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} \varphi^T(0) \\ \vdots \\ \varphi^T(N-1) \end{bmatrix} \theta + \begin{bmatrix} v(0) \\ \vdots \\ v(N-1) \end{bmatrix},$$

But this was worked out for the noise-free case. So we really have,

$$\varphi_0^T(k) = \begin{bmatrix} -y_0(k-1) & \cdots & -y_0(k-n) & u(k-1) & \cdots & u(k-m) \end{bmatrix}$$

And by stacking,

$Y = \Phi_0 \, \theta + V.$ What happens if we use $Y = \Phi \, \theta + V$?

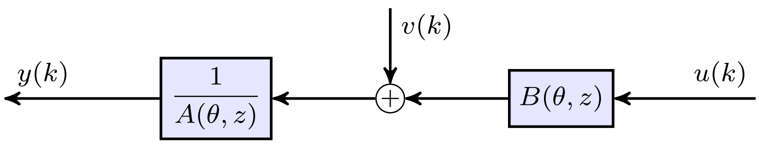
Equation error models

Model form

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = b_1 u(k-1) + \cdots + b_m u(k-m) + v(k)$$

$$A(z)Y(z) = B(z)U(z) + V(z)$$

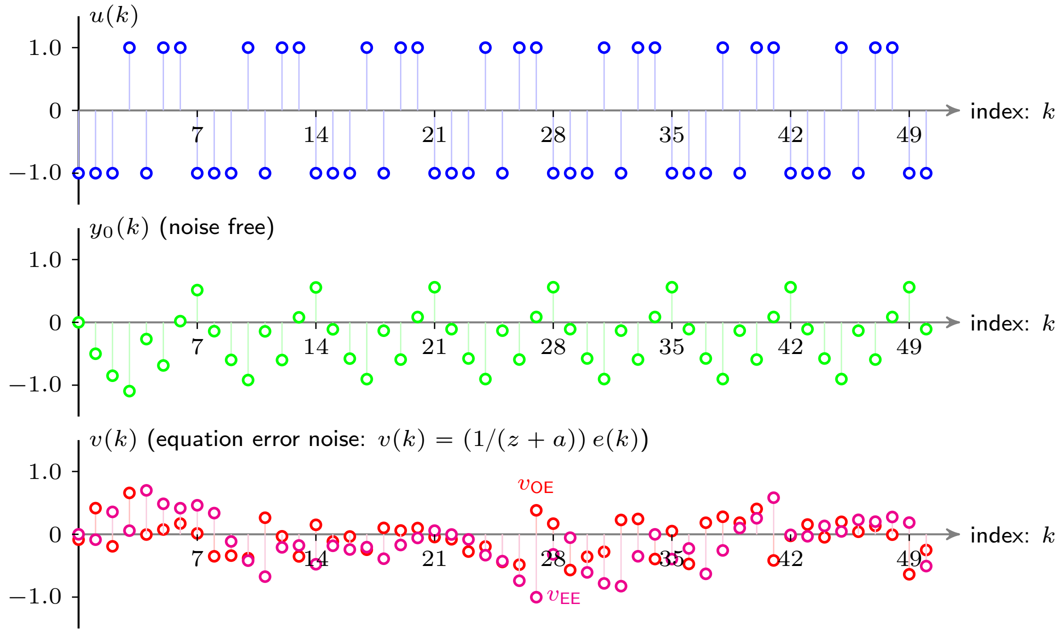
Now we have: $Y = \Phi\theta + V.$



The noise also has the autoregressive dynamics (in $A(z)$).

Example

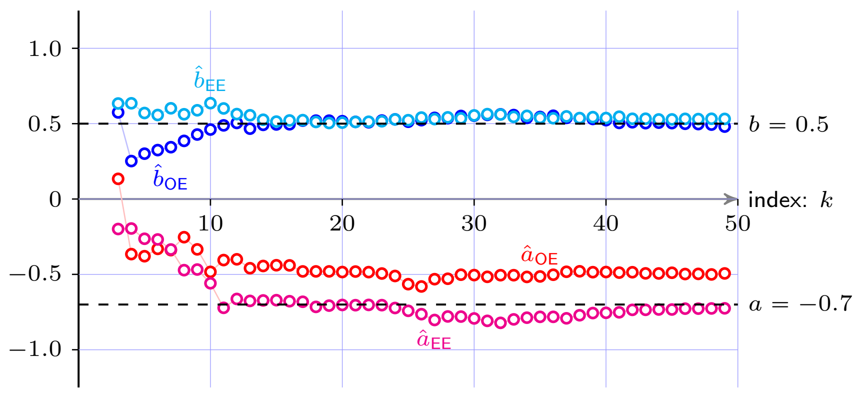
Equation error experiment



$\mathrm{var}(e(k)) = 0.25^2$

Example

Least-squares estimates: equation error structure



True system

$$G(z) = \frac{bz^{-1}}{1 + az^{-1}}, \quad H(z) = \frac{1}{1 + az^{-1}}.$$

Estimation:

$$y(k) \;=\; -\hat{a}_{\mathsf{EE}} y(k-1) \;+\; \hat{b}_{\mathsf{EE}} u(k-1)$$

Output error and Equation error models

Output error framework

Equation error framework

Bibliography

Prediction
Lennart Ljung, *System Identification;Theory for the User*, 2nd Ed., Prentice-Hall, 1999, [section 3.2].

ARX models
Lennart Ljung, *System Identification;Theory for the User*, 2nd Ed., Prentice-Hall, 1999, [sections 1.3 and 4.2].