System Identification

Supplementary notes: lecture 7

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7 Pulse response estimation

7.1 Pulse response estimation

The input-output relationship can be written in terms of the pulse response,

$$y(k) \;=\; \sum_{\tau=0}^{\infty} g(\tau)\, u(k-\tau),$$

where we have assumed a causal system and so $g(\tau) = 0$ for $\tau < 0$. The objective is to use $u(k)$ and $y(k)$ to directly estimate $g(\tau)$.

For a sequence of output values, $y(k)$, $k = 0, \ldots, K-1$, the convolution can be expressed as an infinite matrix equation,

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(K-1) \end{bmatrix} = \begin{bmatrix} u(0) & u(-1) & u(-2) & u(-3) & \cdots \\ u(1) & u(0) & u(-1) & u(-2) & \cdots \\ \vdots & \ddots & \ddots & & \\ u(K-1) & u(K-2) & \cdots & u(0) & \cdots \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ \vdots \end{bmatrix}.$$

The matrix of u values is Toeplitz.

Let's assume that our plant, $g(\tau)$, is stable. This is equivalent to the condition that the pulse response is absolutely summable,

$$\sum_{\tau=0}^{\infty} |g(\tau)| \;<\; \infty.$$

This is trivially satisfied for what are known as finite impulse response (FIR) plants, $g(\tau) = 0$ for all $\tau > \tau_{\max}$. Real rational transfer functions are not FIR but for stable $g(\tau)$ with all poles strictly inside the unit circle we can get arbitrarily close to this case. Given any arbitrarily small $\epsilon > 0$, we can find a value of $\tau_{\max}$ such that,

$$\sum_{\tau=\tau_{\max}}^{\infty} |g(\tau)| \;<\; \epsilon/\tau_{\max}.$$

To see why this is the case note that the assumption on the poles is equivalent to the existence of β and ρ such that,

$$|g(\tau)| < \beta\rho^{\tau}, \quad \beta > 0, \quad 0 < \rho < 1.$$

Now

$$\sum_{\tau=\tau_{\max}}^{\infty} |g(\tau)| \; < \; \beta\rho^{\tau_{\max}} \sum_{i=0}^{\infty} \rho^{i} \; = \; \frac{\beta\rho^{\tau_{\max}}}{1-\rho} \; = \; \frac{\tau_{\max}\beta\rho^{\tau_{\max}}}{(1-\rho)\tau_{\max}}.$$

Now define

$$\epsilon \; = \; \frac{\tau_{\max}\beta\rho^{\tau_{\max}}}{(1-\rho)},$$

and note that because $\rho^{\tau_{\max}}$ decays to zero faster than $\tau_{\max}$ grows we can make ϵ as small as we wish by choosing a sufficiently large $\tau_{\max}$.

Now we can rewrite the convolution as a finite matrix multiplication to within an accuracy of ϵ.

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(K-1) \end{bmatrix} = \begin{bmatrix} u(0) & u(-1) & u(-2) & \cdots & u(-\tau_{\max}) \\ u(1) & u(0) & u(-1) & \cdots & u(1-\tau_{\max}) \\ \vdots & \ddots & \ddots & & \\ u(K-1) & u(K-2) & \cdots & & u(K-\tau_{\max}) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(\tau_{\max}) \end{bmatrix} + \; \epsilon \, \|u\|_{\infty} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

It most practical cases it is easy to obtain a bound on $\|u\|_{\infty}$ and so choose $\tau_{\max}$ to make the finite matrix equation as accurate as we wish. Note that the bound given here is typically very conservative as it assumes the worst case $u(k)$ for $k < 0$.