System Identification

Lecture 7: Pulse response estimation & Persistency of excitation

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Pulse response estimation

Input-output relationship

$$y(k) = \sum_{i=0}^{\infty} g(i)u(k-i) + v(k), \qquad k \;=\; 0, \ldots, K-1.$$

Pulse response estimation

Convolution: matrix formulation

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(K-1) \end{bmatrix} = \begin{bmatrix} u(0) & u(-1) & \cdots & u(-K+1) & \cdots \\ u(1) & u(0) & \cdots & u(-K+2) & \cdots \\ \vdots & & \ddots & \vdots & \\ u(K-1) & u(K-2) & \cdots & u(0) & \cdots \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ \vdots \end{bmatrix}.$$

Pulse response estimation

FIR systems

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(K-1) \end{bmatrix} = \begin{bmatrix} u(0) & u(-1) & \cdots & u(-\tau_{\mathsf{max}}) \\ u(1) & u(0) & \cdots & u(-\tau_{\mathsf{max}}+1) \\ \vdots & & \ddots & \vdots \\ u(K-1) & u(K-2) & \cdots & u(K-\tau_{\mathsf{max}}-1) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(\tau_{\mathsf{max}}) \end{bmatrix}.$$

Pulse response estimation

Strictly stable real-rational systems

If all of the poles of g are strictly inside the unit-circle, then, for any $\epsilon > 0$, we can find τ_{max} such that,

$$\sum_{i=\tau_{\text{max}}+1}^{\infty} |g(i)| \; < \; \epsilon.$$

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(K-1) \end{bmatrix} = \begin{bmatrix} u(0) & u(-1) & \cdots & u(-\tau_{\text{max}}) \\ u(1) & u(0) & \cdots & u(-\tau_{\text{max}}+1) \\ \vdots & & \ddots & \vdots \\ u(K-1) & u(K-2) & \cdots & u(K-\tau_{\text{max}}-1) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(\tau_{\text{max}}) \end{bmatrix} + \begin{bmatrix} e(0) \\ e(1) \\ \vdots \\ e(K-1) \end{bmatrix}$$

$$\|e\|_{\infty} \; \leqslant \; \epsilon \, \|u\|_{\infty}.$$

Pulse response estimation

Initial conditions

If $u(k) = 0$ for all $k < 0$ (system "at rest"),

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(K-1) \end{bmatrix} = \begin{bmatrix} u(0) & 0 & \cdots & 0 \\ u(1) & u(0) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ u(K-1) & u(K-2) & \cdots & u(K-\tau_{\text{max}}-1) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(\tau_{\text{max}}) \end{bmatrix} + \begin{bmatrix} e(0) \\ e(1) \\ \vdots \\ e(K-1) \end{bmatrix}$$

$$\|e\|_{\infty} \; \leqslant \; \epsilon \, \|u\|_{\infty}.$$

Pulse response estimation

Initial conditions

If $u(k)$ is unknown for $k < 0$, but bounded ($\|u(k)\|_\infty$),

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(\tau_{\text{max}}) \\ \vdots \\ y(K-1) \end{bmatrix} = \begin{bmatrix} u(0) & 0 & \cdots & 0 \\ u(1) & u(0) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ u(\tau_{\text{max}}) & \cdots & & u(0) \\ \vdots & & & \vdots \\ u(K-1) & u(K-2) & \cdots & u(K-\tau_{\text{max}}-1) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(\tau_{\text{max}}) \end{bmatrix} + \begin{bmatrix} e(0) \\ e(1) \\ \vdots \\ e(\tau_{\text{max}}) \\ \vdots \\ e(K-1) \end{bmatrix}$$

$\|e\|_\infty \leqslant \epsilon \|u\|_\infty$ for all $k > \tau_{\text{max}}$.

Pulse response estimation

Bad measurements

Corrupted measurement: $y(j)$

$$\begin{bmatrix} y(\tau_{\text{max}}) \\ \vdots \\ y(j-1) \\ y(j) \\ y(j+1) \\ \vdots \\ y(K-1) \end{bmatrix} = \begin{bmatrix} u(\tau_{\text{max}}) & \cdots & & u(0) \\ \vdots & & & \vdots \\ u(j-1) & u(j-2) & \cdots & u(j-\tau_{\text{max}}-2) \\ u(j) & u(j-1) & \cdots & u(j-\tau_{\text{max}}-1) \\ u(j+1) & u(j) & \cdots & u(j-\tau_{\text{max}}) \\ \vdots & & & \vdots \\ u(K-1) & u(K-2) & \cdots & u(K-\tau_{\text{max}}+1) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(\tau_{\text{max}}) \end{bmatrix} + \begin{bmatrix} e(\tau_{\text{max}}) \\ \vdots \\ e(j-1) \\ e(j) \\ e(j+1) \\ \vdots \\ e(K-1) \end{bmatrix}$$

$\|e\|_\infty \leqslant \epsilon \|u\|_\infty$ for all $k > \tau_{\text{max}}$.

Pulse response estimation

General formulation

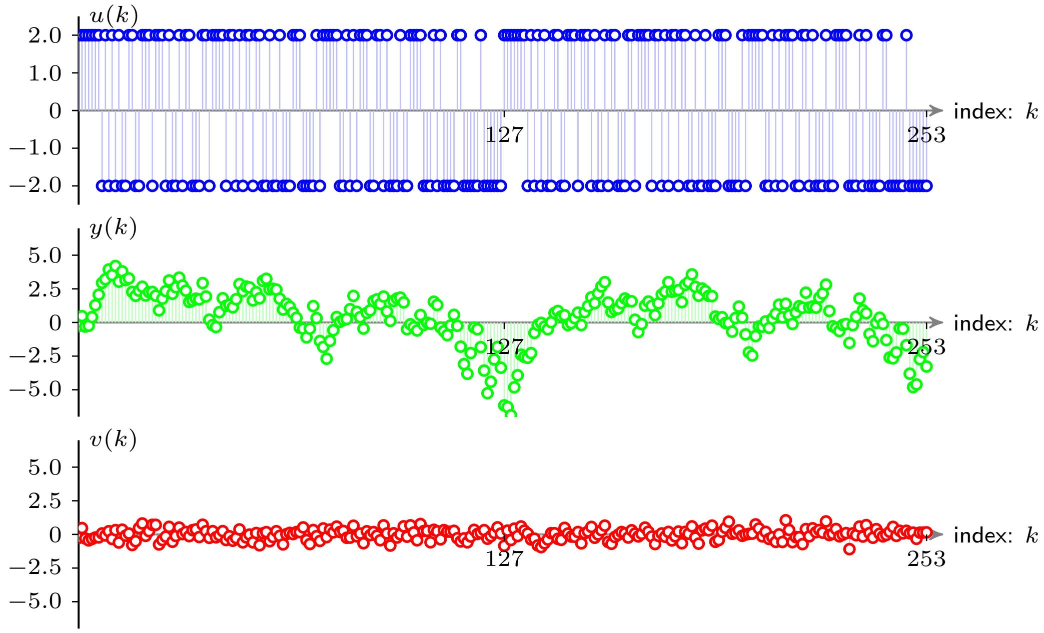
$$y = \Phi_u g + e, \qquad y \in \mathcal{R}^K,\ \Phi_u \in \mathcal{R}^{K \times \tau_{\max}+1}, g \in \mathcal{R}^{\tau_{\max}+1}, e \in \mathcal{R}^K.$$

Noisy measurements

$$y = \Phi_u g + e + v, \qquad y \in \mathcal{R}^K,\ \Phi_u \in \mathcal{R}^{K \times \tau_{\max}+1}, g \in \mathcal{R}^{\tau_{\max}+1}, e \in \mathcal{R}^K, v \in \mathcal{R}^K.$$

Normal equations

Example



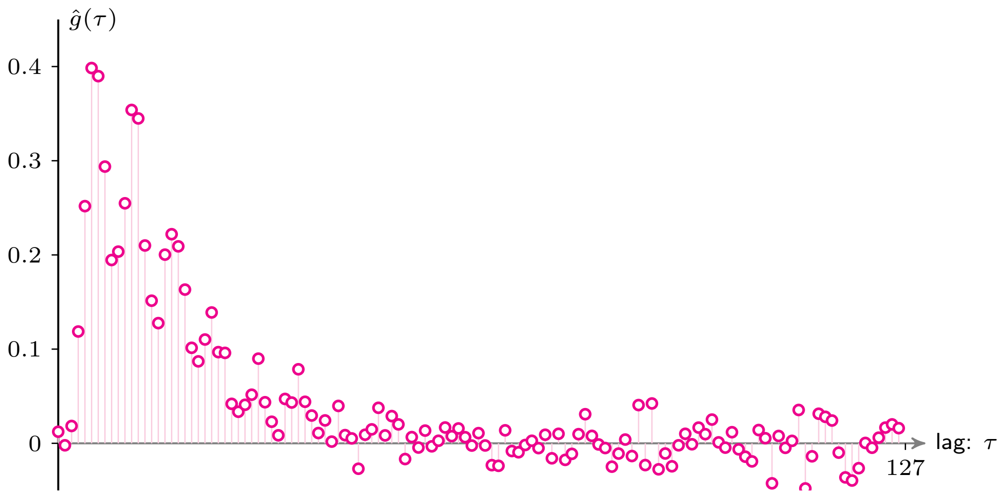
Two periods of a 127-length PRBS signal. Initial conditions: $u(k) = 0$ for $k < 0$.

Example

Problem configuration

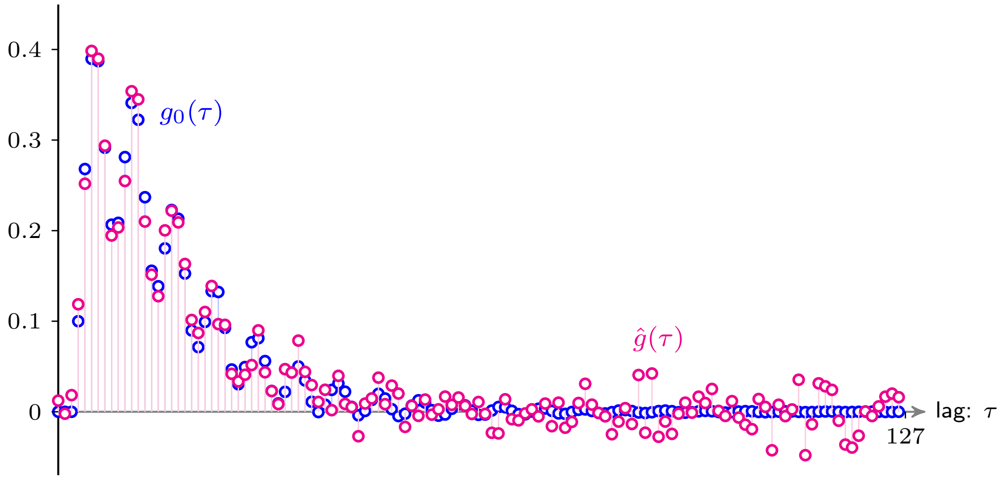
Example

Pulse response estimate



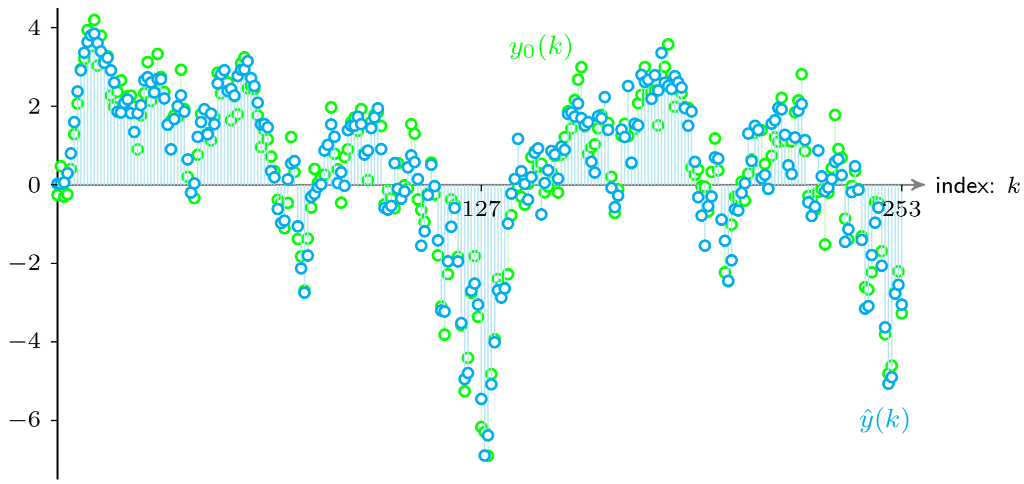
Example

Estimate $\hat{g}(k)$ and true system $g_0(k)$



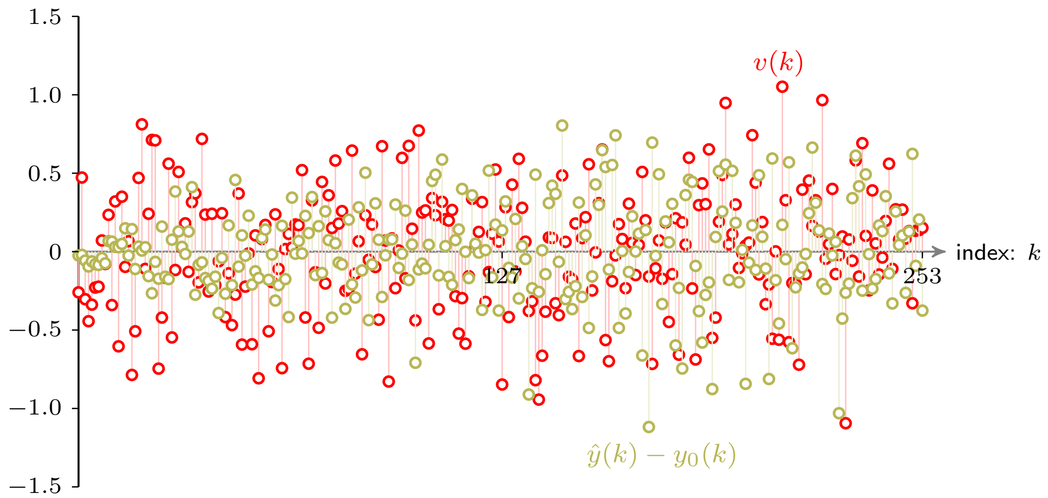
Example

Estimated output signal comparison



Example

Estimated output signal error: $\hat{y}(k) - y_0(k)$



$$\|v(k)\|_2 \;=\; 6.29, \qquad \|\hat{y}(k) - y_0(k)\|_2 = 5.080.$$

Pulse response estimation

Input-output relationship

$$y(k) = \sum_{i=0}^{\infty} g(i)u(k-i) + v(k)$$

$$\begin{aligned} E\{y(k)u(k-\tau)\} &= E\left\{\sum_{i=0}^{\infty} g(i)u(k-i)u(k-\tau)\right\} + E\{v(k)u(k-\tau)\} \\ &= \sum_{i=0}^{\infty} g(i)E\{u(k-i)u(k-\tau)\} \quad \text{(if noise is uncorrelated)} \\ &= \sum_{i=0}^{\infty} g(i)R_u(\tau-i) \end{aligned}$$

$$R_{yu}(\tau) = g(i) \ast R_u(\tau). \qquad \text{(convolution)}$$

Pulse response estimation

Matrix representation

The input-output relationship,

$$R_{yu}(\tau) = g(\tau) \ast R_u(\tau)$$

can be written in matrix form,

$$\begin{bmatrix} R_{yu}(0) \\ R_{yu}(1) \\ R_{yu}(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} R_u(0) & R_u(-1) & R_u(-2) & \cdots \\ R_u(1) & R_u(0) & R_u(-1) & \\ R_u(2) & R_u(1) & \ddots & \ddots \\ \vdots & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ \vdots \end{bmatrix}.$$

Pulse response estimation

Finite data estimate

Create finite length estimates from this relationship;

$$\underbrace{\begin{bmatrix} \hat{R}_{yu}(0) \\ \vdots \\ \hat{R}_{yu}(K-1) \end{bmatrix}}_{\hat{R}_{yu}} = \underbrace{\begin{bmatrix} \hat{R}_u(0) & \cdots & \hat{R}_u(-(K-1)) \\ \vdots & & \vdots \\ \hat{R}_u(K-1) & \cdots & \hat{R}_u(0) \end{bmatrix}}_{\bar{R}_K} \begin{bmatrix} \hat{g}(0) \\ \vdots \\ \hat{g}(K-1) \end{bmatrix}.$$

Note that $R_u(-\tau) = R_u(\tau)$.

In the periodic (noise-free) and FIR case this is exact,

if K is the period and $g(\tau) = 0$ for $\tau \geqslant K \geqslant \tau_{\mathsf{max}}$.

Pulse response estimation

If $\bar{R}_K$ is invertible

$$\hat{g} = \bar{R}_K^{-1} \hat{R}_{yu}.$$

Invertibility of $\bar{R}_K$ is guaranteed if $u(k)$ is "persistently exciting".

Under these conditions $\hat{g}$ is uniquely determined.

Pulse response estimation

Bias

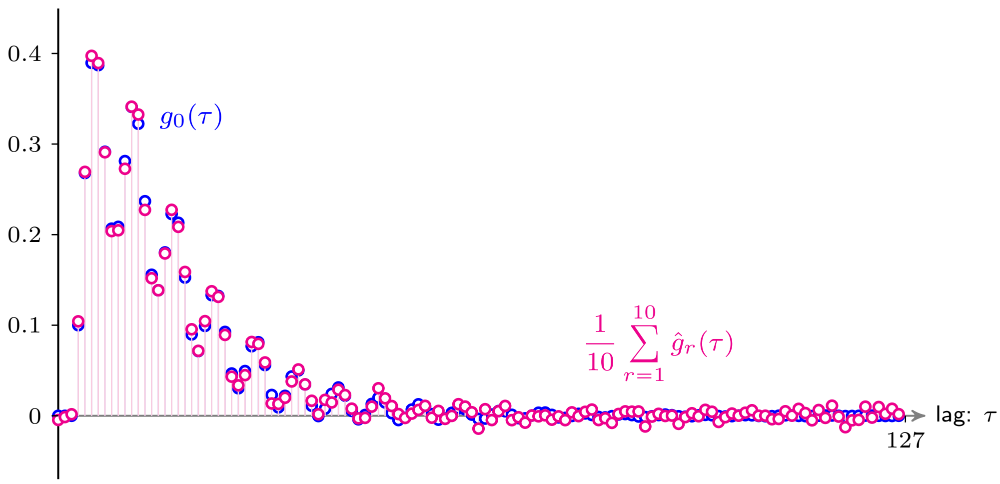
Variance

Simple noise model: $v(k) \sim \mathcal{N}(0, \sigma_v^2)$. (i.i.d.)

$$\operatorname{cov}(\hat{g}) = \mathcal{E}\left\{(\hat{g} - \mathcal{E}\{\hat{g}\})(\hat{g} - \mathcal{E}\{\hat{g}\})^T\right\} = \sigma_v^2 (\Phi_u^T \Phi_u)^{-1}.$$

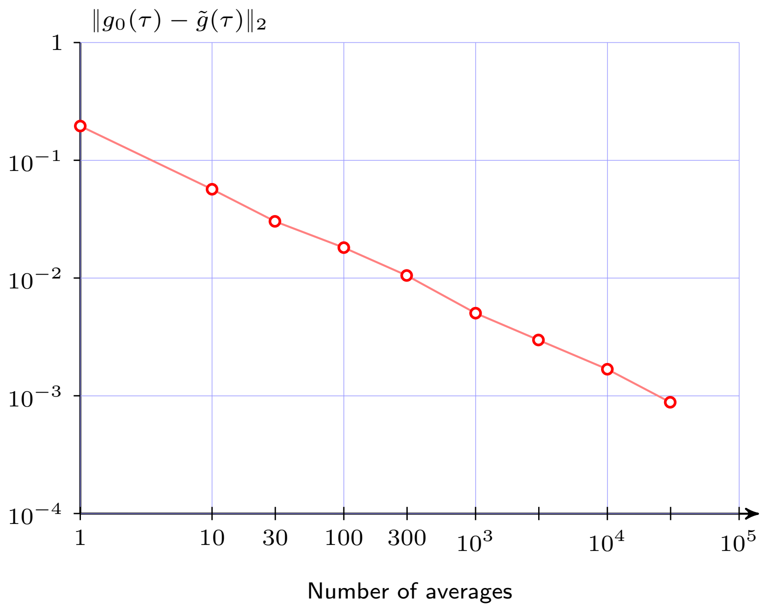
Example

Averaged estimates (10)



Example

Convergence of the estimates



Pulse response estimation

Asymptotically uncorrelated pulse response estimates

Asymptotically uncorrelated if:

$\Phi_u^T \Phi_u$ is diagonal.

Cases:

- $u(k) \sim \mathcal{N}(0, \sigma_u^2)$ (white noise)
- u is a PRBS signal (only approximately).

Persistency of excitation

A stationary input, $u(k)$, is persistently exciting of order N if,

$$\bar{R}_N = \begin{bmatrix} R_u(0) & \cdots & R_u(-(N-1)) \\ \vdots & & \vdots \\ R_u(N-1) & \cdots & R_u(0) \end{bmatrix}$$

is positive definite ($\bar{R}_N > 0$).

This is sufficient to uniquely determine the first N coefficients of the pulse response, $\hat{g}(k)$.

The definition also applies to deterministic (or quasi-stationary) signals.

A signal is simply called persistently exciting if this holds for all N.

Persistency of excitation examples

Step function

$$u(k) = 1, \quad k = 0, 1, \ldots \qquad \text{is persistently exciting of order 1.}$$

PRBS signal of period M:

This is persistently exciting of order M.

Sum of sinusoids:

$$u(k) = \sum_{s=1}^{S} \alpha_s \cos(\omega_s k + \phi_s)$$

$$\text{persistently exciting of order } \begin{cases} 2S & \text{if } 0 < \omega_s < \pi, \;\; s = 1, \ldots, S \\ 2S - 1 & \text{if } \omega = 0 \text{ or } \omega = \pi \in \{\omega_s, s = 1, \ldots, S\} \\ 2S - 2 & \text{if } \omega = 0 \text{ and } \omega = \pi \in \{\omega_s, s = 1, \ldots, S\} \end{cases}$$

Persistency of excitation in pulse response estimation

Bibliography

Persistency of excitation

Lennart Ljung, *System Identification;Theory for the User*, 2nd Ed., Prentice-Hall, 1999, [section 13.2].