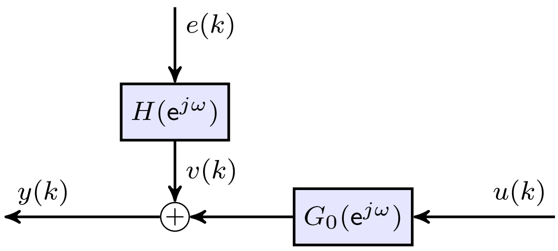
System Identification

Lecture 5: Frequency-domain identification

Roy Smith

Frequency domain models



Model assumptions:

Linear, time-invariant system, $g_0(l)$:

$$y(k) \;=\; \sum_{l=0}^{\infty} g_0(l)u(k-l) \;+\; v(k), \quad k = 0, 1, \ldots$$

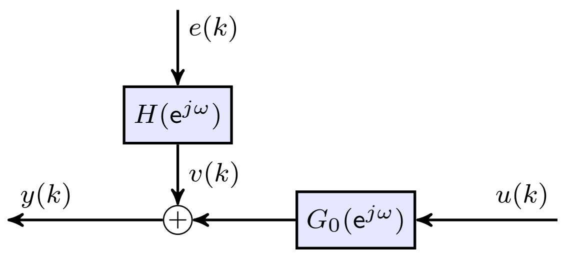
Assumptions: causal: $\iff$ $g_0(l) = 0$, for all $l < 0$.

noise: $E\{v(k)\} = 0$, (zero-mean)
stationary.

Frequency domain estimation

$$Y(\mathrm{e}^{j\omega}) \;=\; G_0(\mathrm{e}^{j\omega})U(\mathrm{e}^{j\omega}) \;+\; V(\mathrm{e}^{j\omega})$$

Swept-sine identification



Fundamental property of linear systems

$$Y(\mathrm{e}^{j\omega}) \;=\; G_0(\mathrm{e}^{j\omega})U(\mathrm{e}^{j\omega}) \;+\; V(\mathrm{e}^{j\omega})$$

Identification approach:

Apply a series of sinusoidal inputs ("swept-sine" input)

and for each find the gain and phase change of the output.

Swept-sine identification

Method:

Select the input frequency, $0 < \omega_u < \pi$ (rad./sample), and calculation length, N, such that,

$$\omega_u = r\frac{2\pi}{N} \quad \text{for some integer } r.$$

Input:

$$u(k) = \alpha\cos(\omega_u k), \quad k = 0, 1, \ldots, K-1 \quad (\text{with } K \geqslant N),$$

Output:

$$y(k) = \alpha\left|G_0(\mathrm{e}^{j\omega_u})\right|\cos(\omega_u k + \theta(\omega_u)) + v(k) + \text{transient}$$

$$\text{where} \quad \theta(\omega_u) = \arg(G_0(\mathrm{e}^{j\omega_u}))$$

Swept-sine identification: correlation analysis

Correlation functions: (calculation length $= N$).

$$I_c(N) = \frac{1}{N}\sum_{k=0}^{N-1} y(k)\cos(\omega_u k)$$

$$I_s(N) = \frac{1}{N}\sum_{k=0}^{N-1} y(k)\sin(\omega_u k)$$

These can be calculated from the data.

Expanding:

$$I_c(N) = \frac{\alpha}{2}\left|G_0(\mathrm{e}^{j\omega_u})\right|\cos(\theta(\omega_u)) + \frac{\alpha}{2}\left|G_0(\mathrm{e}^{j\omega_u})\right|\frac{1}{N}\sum_{k=0}^{N-1}\cos(2\omega_u k + \theta(\omega_u)) + \frac{1}{N}\sum_{k=0}^{N-1} v(k)\cos(\omega_u k)$$

Swept-sine identification: correlation analysis

If the noise, $v(k)$, is sufficiently uncorrelated

(for example, is filtered gaussian noise),

then the variance satisfies,

$$\lim_{N \longrightarrow \infty} \operatorname{var}\left(\frac{1}{N} \sum_{k=0}^{N-1} v(k) \cos(\omega_u k) \right) \; = \; 0$$

with a convergence rate of $1/N$.

Swept-sine identification: correlation analysis

So, as in the limit as $N \longrightarrow \infty$,

$$E\{I_c(N)\} \longrightarrow \frac{\alpha}{2} \left| G_0(\mathrm{e}^{j\omega_u}) \right| \cos(\theta(\omega_u))$$

$$E\{I_s(N)\} \longrightarrow \frac{-\alpha}{2} \left| G_0(\mathrm{e}^{j\omega_u}) \right| \sin(\theta(\omega_u))$$

and $\lim_{N \longrightarrow \infty} \operatorname{var}(I_c(N)) = 0, \quad \lim_{N \longrightarrow \infty} \operatorname{var}(I_s(N)) = 0$

Estimate the transfer function via:

$$\hat{G}_N(\mathrm{e}^{j\omega_u}) = \frac{I_c(N) - jI_s(N)}{\alpha/2},$$

or equivalently, the gain and phase:

$$\left| \hat{G}_N(\mathrm{e}^{j\omega_u}) \right| = \frac{\sqrt{I_c(N)^2 + I_s(N)^2}}{\alpha/2}$$

$$\arg \hat{G}_N(\mathrm{e}^{j\omega_u}) = -\arctan \frac{I_s(N)}{I_c(N)}$$

Sweptsine ID methods

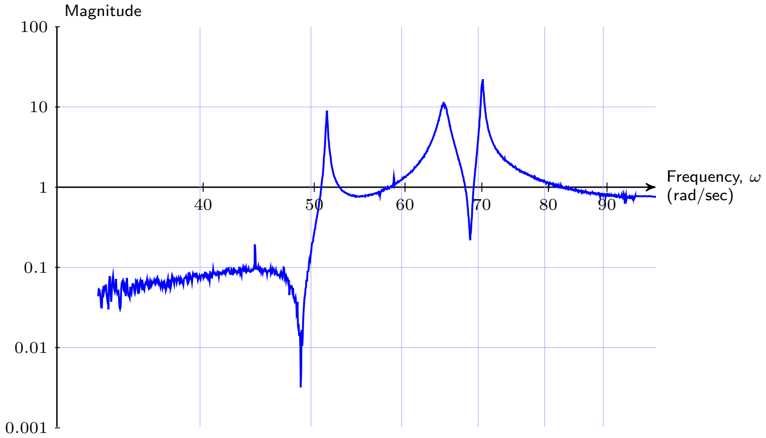
Advantages:

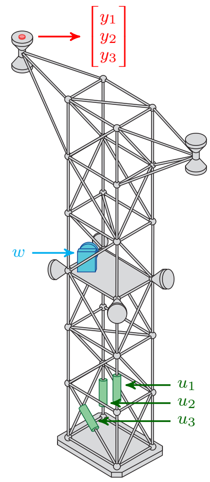
- Energy is concentrated at the frequencies of interest.
- Amplitude of $u(k)$ can easily be tuned as a function of frequency.
- Easy to avoid saturation and tune signal/noise (S/N) ratio.

Disadvantages:

- A large amount of data is required.
- Significant amount of time required for experiments.
- Some processes won't allow sinusoidal inputs.

Swept-sine example: flexible structure





Experiments:

Anti-aliasing filters:	single pole, 1.0 kHz cutoff	
1600 sinusoids:	32.20 rad/sec. to 95.01 rad/sec.	$\Delta\omega$ = 0.0393 rad/sec.
	63.81 rad/sec. to 692.13 rad/sec.	$\Delta\omega$ = 0.393 rad/sec.

Analysis:

DFTs	4086 length records	25 averages per sinusoid

Empirical transfer function estimation (ETFE)

Approximation:

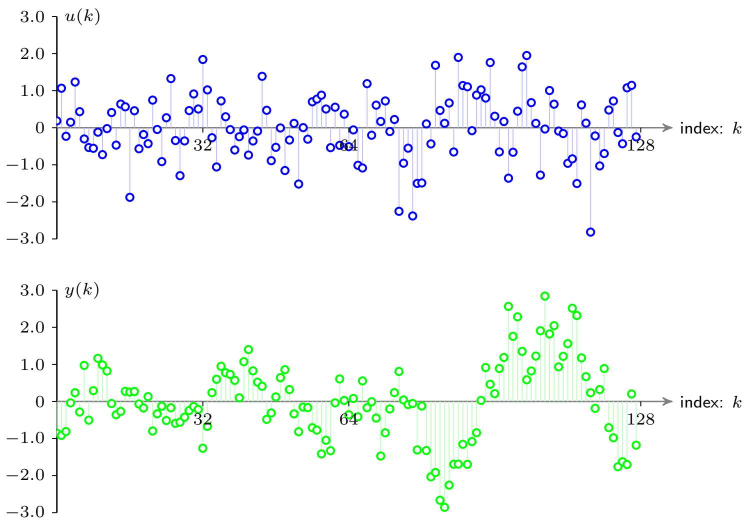
$$\underbrace{Y_N(\mathrm{e}^{j\omega_n})}_{\text{length-N DFT}} = \sum_{k=0}^{N-1} y(k)\mathrm{e}^{-j\omega_n k} \approx \sum_{k=-\infty}^{\infty} y(k)\mathrm{e}^{-j\omega_n k} = Y(\mathrm{e}^{j\omega_n})$$

$$\underbrace{U_N(\mathrm{e}^{j\omega_n})}_{\text{length-N DFT}} = \sum_{k=0}^{N-1} u(k)\mathrm{e}^{-j\omega_n k} \approx \sum_{k=-\infty}^{\infty} u(k)\mathrm{e}^{-j\omega_n k} = U(\mathrm{e}^{j\omega_n})$$

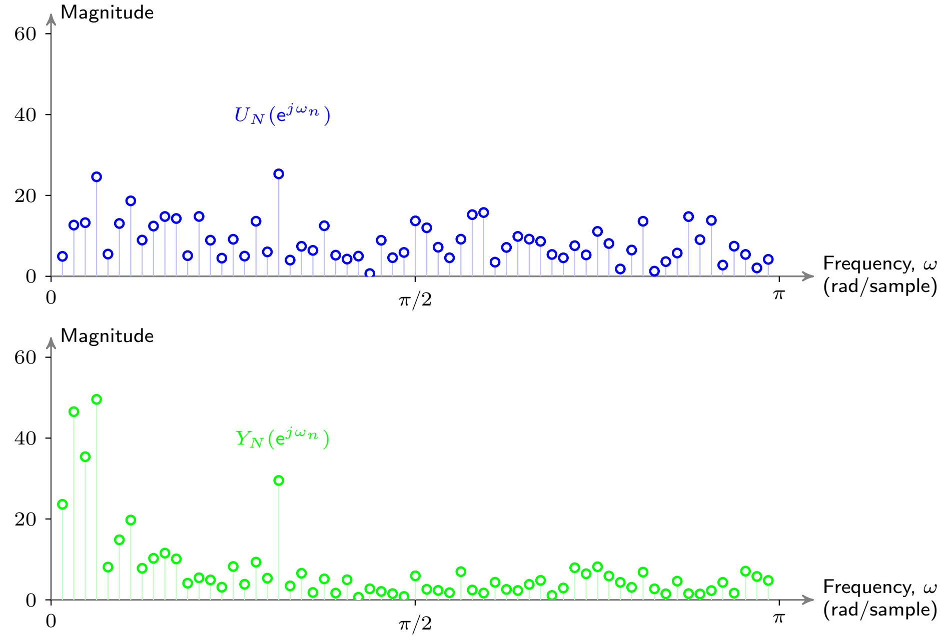
$$\underbrace{\hat{G}_N(\mathrm{e}^{j\omega_n})}_{\text{ETFE}} := \frac{Y_N(\mathrm{e}^{j\omega_n})}{U_N(\mathrm{e}^{j\omega_n})}$$

One choice for N is $N = K$ (data length).

ETFE example: “experimental” data

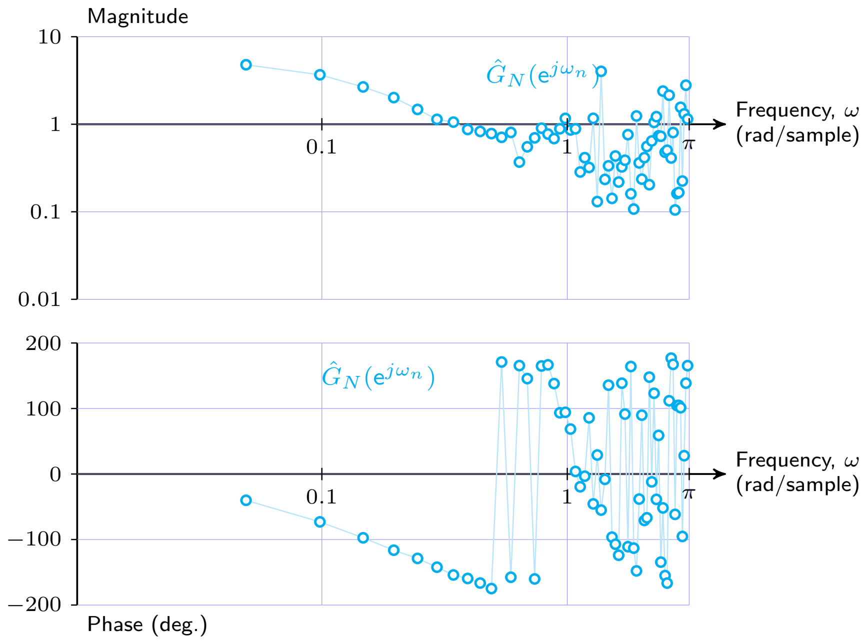


ETFE example: FFT calculation results ($N = 128$)



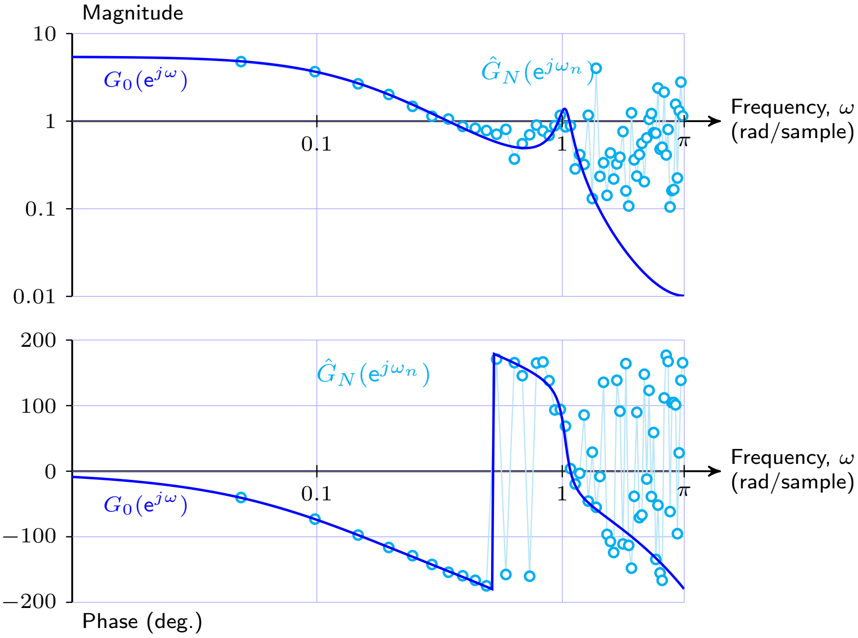
ETFE example

$$\hat{G}_N(e^{j\omega_n}) = Y_N(e^{j\omega_n})/U_N(e^{j\omega_n})$$



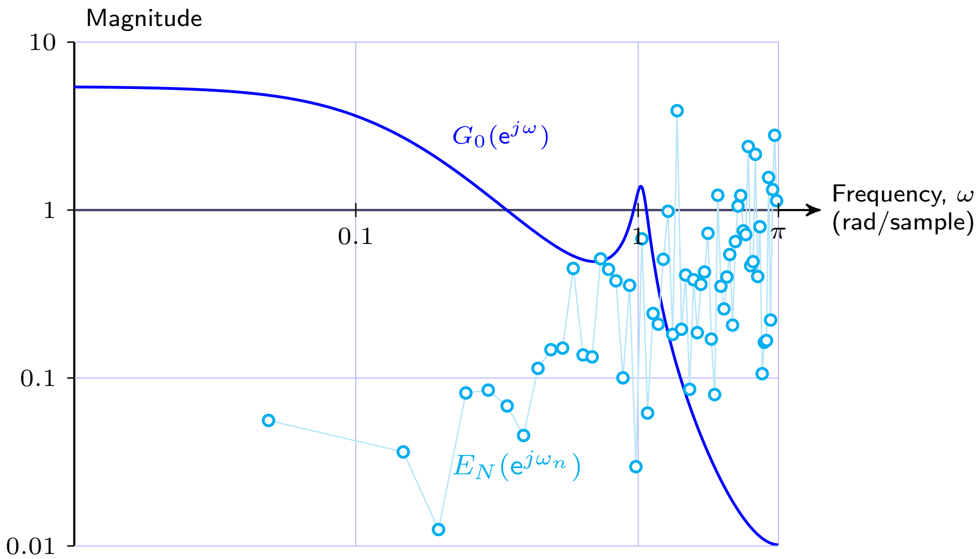
ETFE example

$$\hat{G}_N(\mathrm{e}^{j\omega_n}) = Y_N(\mathrm{e}^{j\omega_n})/U_N(\mathrm{e}^{j\omega_n})$$



ETFE example: absolute error

$$E_N(\mathrm{e}^{j\omega_n}) = G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})$$



ETFE example: MATLAB calculations

```
U = fft(u);                           % calculate N point FFTs
Y = fft(y);
omega = (2*pi/N)*(0:N-1)';            % frequency grid
idx = find(omega > 0 & omega < pi);   % positive frequencies
loglog(omega(idx),abs(U(idx)))
loglog(omega(idx),abs(Y(idx)))

Gest = Y./U;                          % ETFE estimate
Gfresp = squeeze(freqresp(G,omega));  % "true" system response
loglog(omega(idx),abs(Gest(idx)))
semilogx(omega(idx),angle(Gest(idx)))

Err = Gest - Gfresp;                  % calculate error
loglog(omega(idx),abs(Err(idx)))
```


Empirical transfer function estimation

Periodic input case:

Empirical transfer function estimation

Periodic input case:

Period M inputs: $u(k) = u(k+M)$:

Then,

$$Y_N(\mathrm{e}^{j\omega_n}) = G_0(\mathrm{e}^{j\omega_n})U_N(\mathrm{e}^{j\omega_n}) + V_N(\mathrm{e}^{j\omega_n})$$

$$\hat{G}_N(\mathrm{e}^{j\omega_n}) = G_0(\mathrm{e}^{j\omega_n}) + \frac{V_N(\mathrm{e}^{j\omega_n})}{U_N(\mathrm{e}^{j\omega_n})}$$

ETFE error properties:

Bias properties

$$\hat{G}_N(\mathrm{e}^{j\omega_n}) = \frac{Y_N(\mathrm{e}^{j\omega_n})}{U_N(\mathrm{e}^{j\omega_n})} = G_0(\mathrm{e}^{j\omega_n}) + \frac{V_N(\mathrm{e}^{j\omega_n})}{U_N(\mathrm{e}^{j\omega_n})}$$

And we find the bias by examining,

$$\begin{aligned} E\{\hat{G}_N(\mathrm{e}^{j\omega_n})\} &= G_0(\mathrm{e}^{j\omega_n}) + E\left\{\frac{V_N(\mathrm{e}^{j\omega_n})}{U_N(\mathrm{e}^{j\omega_n})}\right\} \\ &= G_0(\mathrm{e}^{j\omega_n}) \quad \text{(assumes zero mean noise)} \end{aligned}$$

For periodic inputs (with N being an integer number of periods):

— the ETFE is unbiased.

ETFE error properties:

Variance properties

Variance (for unbiased case $E\{\hat{G}_N(e^{j\omega_n})\} = G_0(e^{j\omega_n})$):

$$E\left\{\left|\hat{G}_N(e^{j\omega_n}) - G_0(e^{j\omega_n})\right|^2\right\} = \frac{\phi_v(e^{j\omega_n}) + \frac{2}{N}c}{\frac{1}{N}|U_N(e^{j\omega_n})|^2},$$

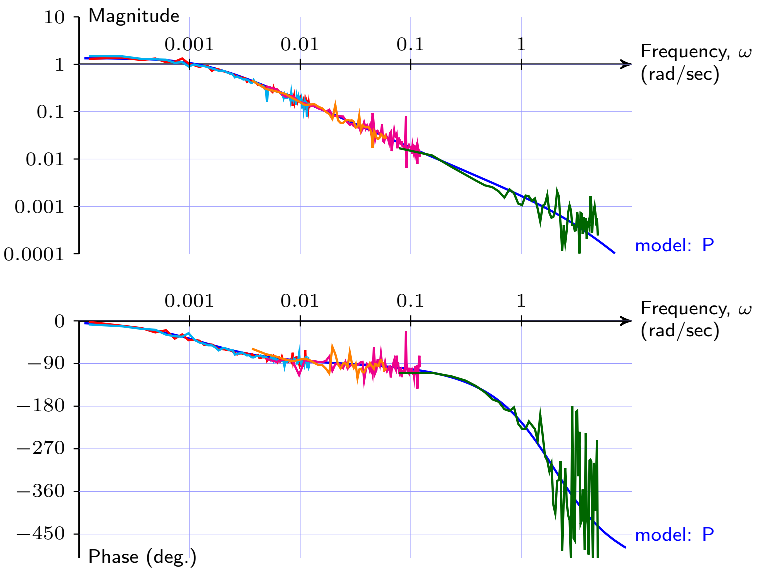
where $|c| \leqslant C = \sum_{\tau=1}^{\infty} |\tau R_v(\tau)|$ is assumed to be finite.

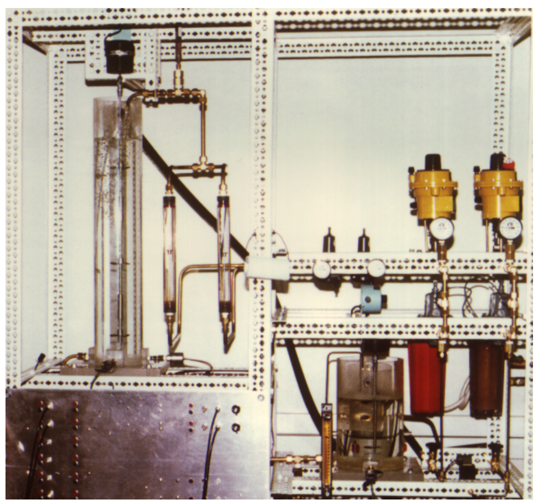
For estimates at different frequencies ($\omega_n \neq \omega_i$):

$$E\left\{\left(\hat{G}_N(e^{j\omega_n}) - G_0(e^{j\omega_n})\right)\left(\hat{G}_N(e^{-j\omega_i}) - G_0(e^{-j\omega_i})\right)\right\} = 0$$

ETFE example: process control

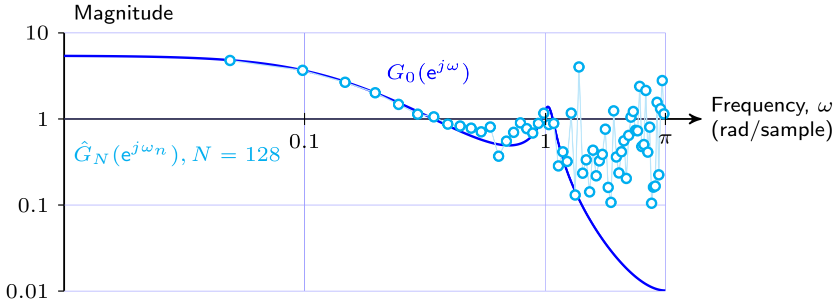
Model: total flow to tank height.

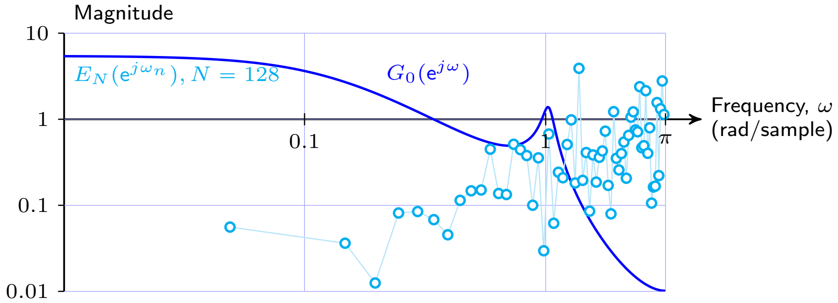




ETFE example — taking more data

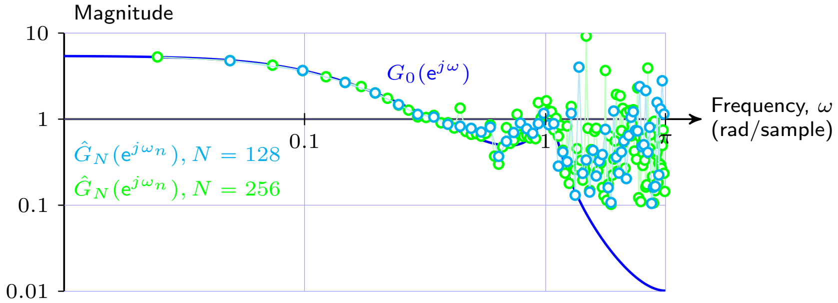
$$E_N(e^{j\omega_n}) = G_0(e^{j\omega_n}) - \hat{G}_N(e^{j\omega_n})$$

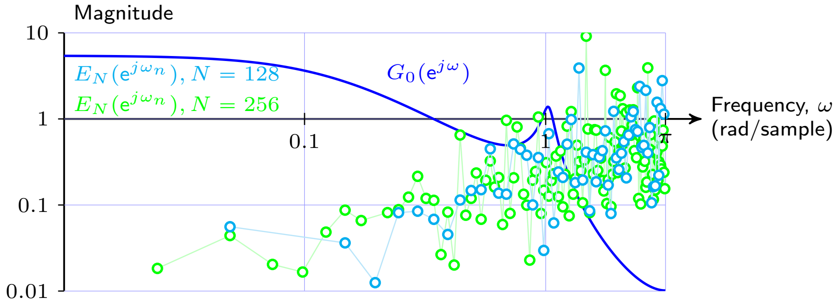




ETFE example — taking more data

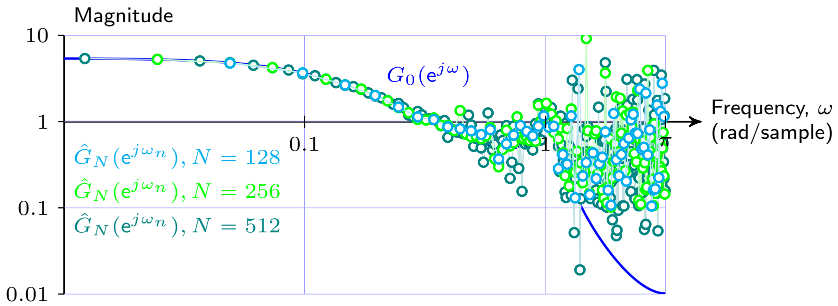
$$E_N(e^{j\omega_n}) = G_0(e^{j\omega_n}) - \hat{G}_N(e^{j\omega_n})$$

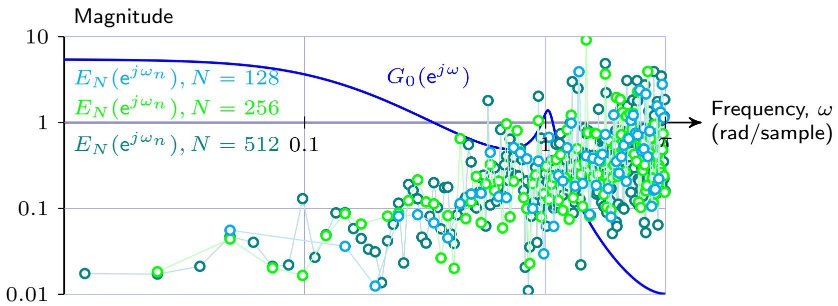




ETFE example — taking more data

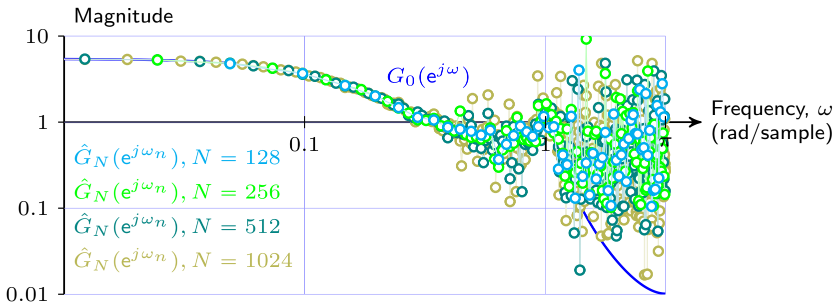
$$E_N(\mathrm{e}^{j\omega_n}) = G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})$$

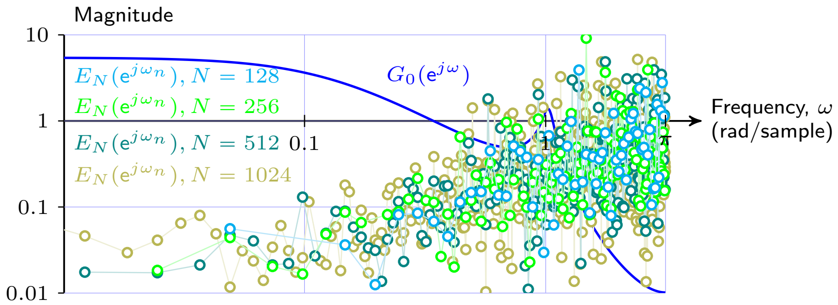




ETFE example — taking more data

$$E_N(\mathrm{e}^{j\omega_n}) = G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})$$

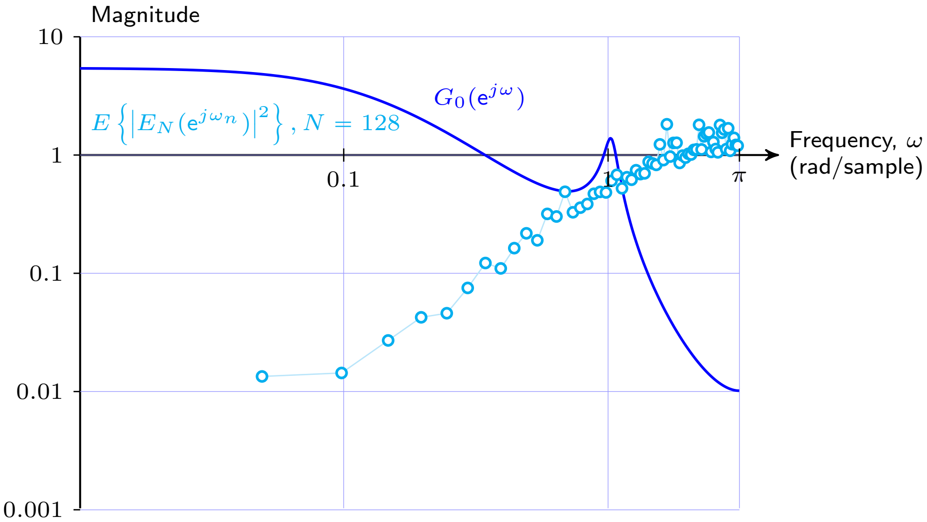




ETFE example: mean-square error

Mean-square error: rerunning the experiments 10,000 times.

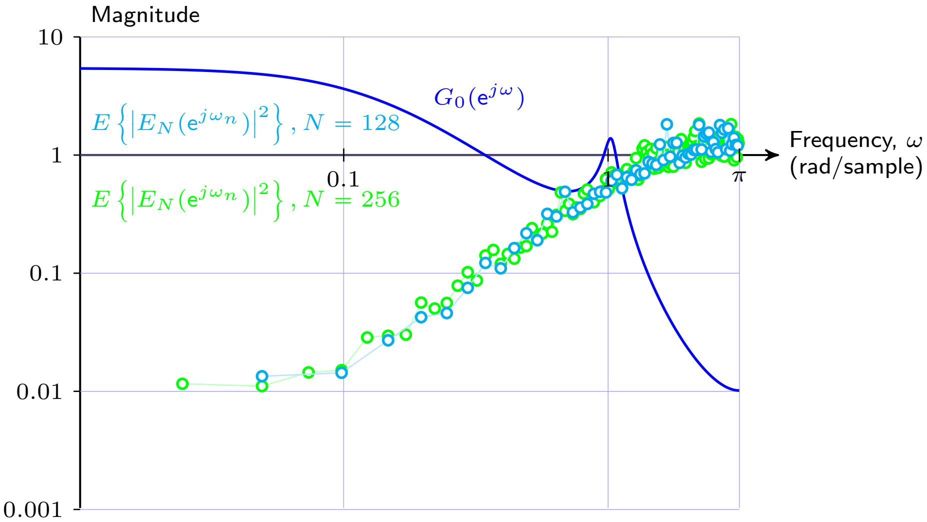
$$E\left\{\left|E_N(\mathrm{e}^{j\omega_n})\right|^2\right\} \approx \frac{1}{10,000}\sum_{l=1}^{10,000}\left|G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})\right|^2$$



ETFE example: mean-square error

Mean-square error: rerunning the experiments 10,000 times.

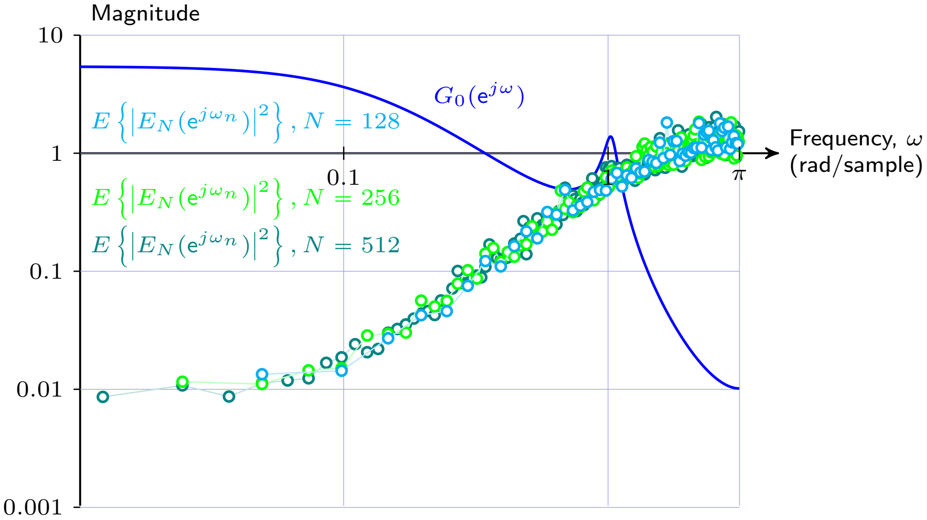
$$E\left\{\left|E_N(\mathrm{e}^{j\omega_n})\right|^2\right\} \approx \frac{1}{10,000}\sum_{l=1}^{10,000}\left|G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})\right|^2$$



ETFE example: mean-square error

Mean-square error: rerunning the experiments 10,000 times.

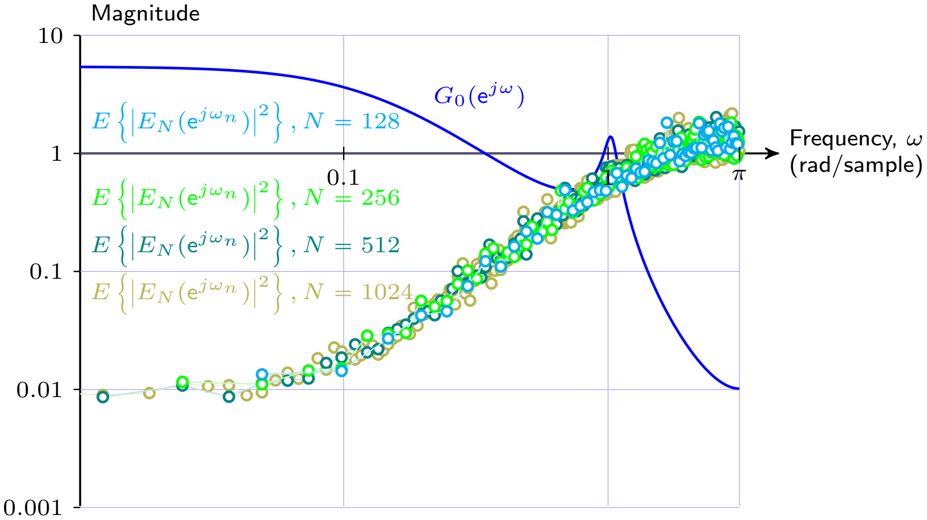
$$E\left\{\left|E_N(\mathrm{e}^{j\omega_n})\right|^2\right\} \approx \frac{1}{10,000}\sum_{l=1}^{10,000}\left|G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})\right|^2$$



ETFE example: mean-square error

Mean-square error: rerunning the experiments 10,000 times.

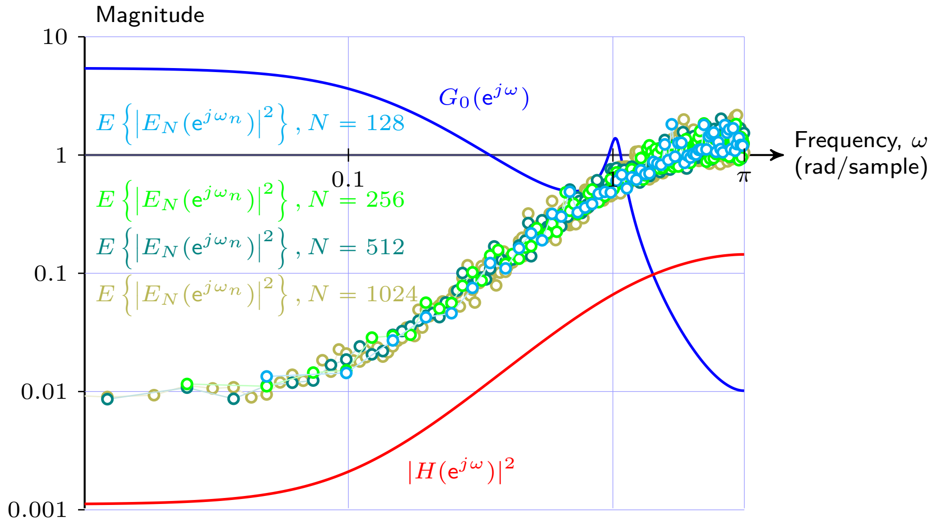
$$E\left\{\left|E_N(\mathrm{e}^{j\omega_n})\right|^2\right\} \approx \frac{1}{10,000}\sum_{l=1}^{10,000}\left|G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})\right|^2$$



ETFE example: mean-square error

Mean-square error: rerunning the experiments 10,000 times.

$$E\left\{\left|E_N(\mathrm{e}^{j\omega_n})\right|^2\right\} \approx \frac{1}{10,000}\sum_{l=1}^{10,000}\left|G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})\right|^2$$



Empirical transfer function estimation

Transient responses:

Initial transient corrupts the measurement:

$$y(k) = G_0\left(u_{\text{periodic}}(k)W_{[0,N-1]}(k)\right) + v(k),$$

with the "window" function:

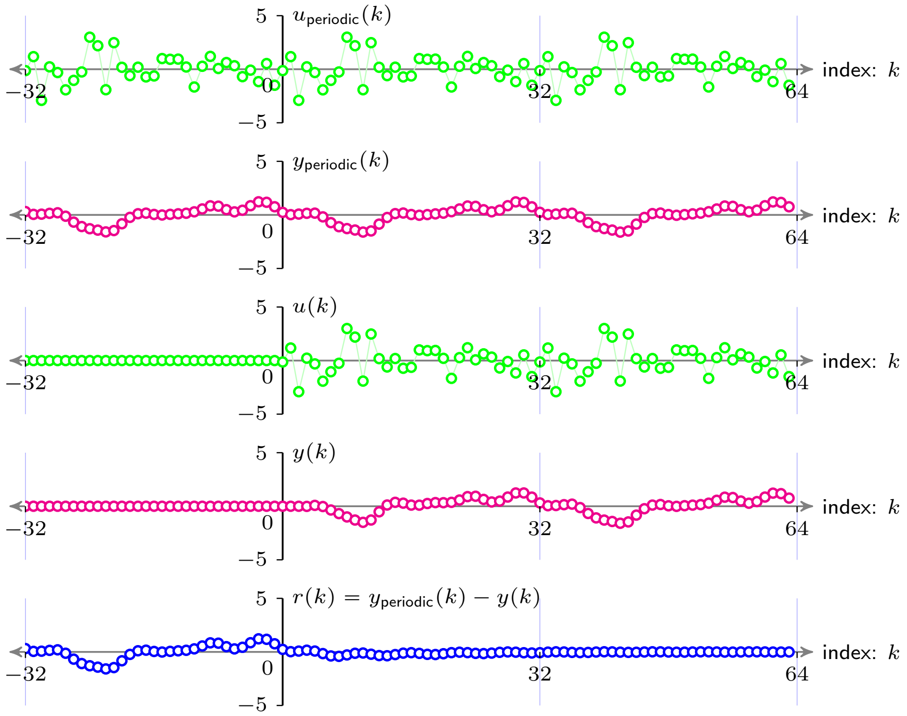
$$W_{[0,N-1]}(k) = \begin{cases} 1 & \text{if } 0 \leqslant k < N \\ 0 & \text{otherwise} \end{cases}$$

For all outputs up to time $k = N - 1$,

$$y(k) = G_0\, u_{\text{periodic}}(k) - \underbrace{G_0\left(u_{\text{periodic}} W_{(-\infty,-1]}\right)}_{r(k)} + v(k)$$

$$Y_N(\mathrm{e}^{j\omega_n}) = G_0(\mathrm{e}^{j\omega_n})U_N(\mathrm{e}^{j\omega_n}) + R_N(\mathrm{e}^{j\omega_n}) + V_N(\mathrm{e}^{j\omega_n})$$

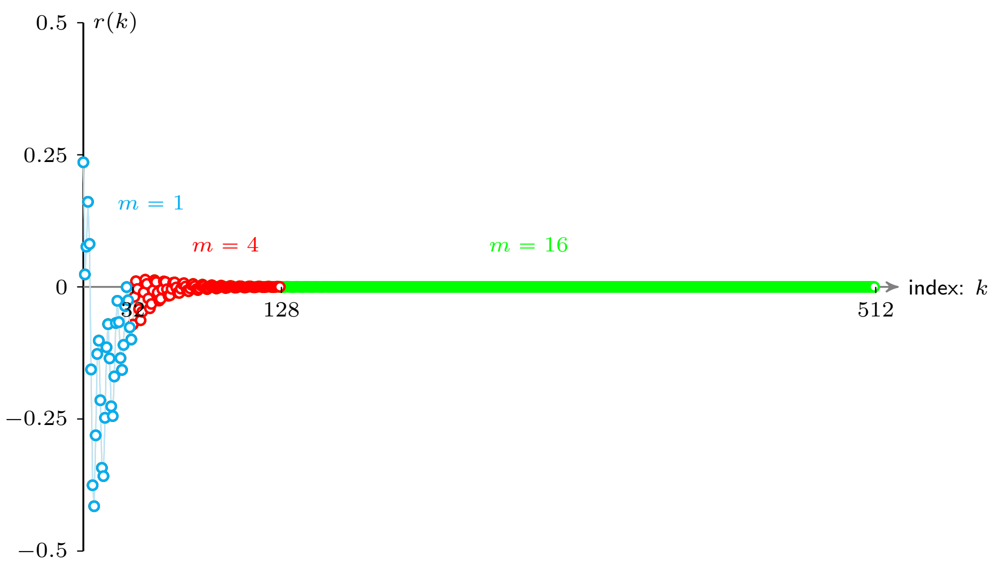
Transient responses



Transient responses

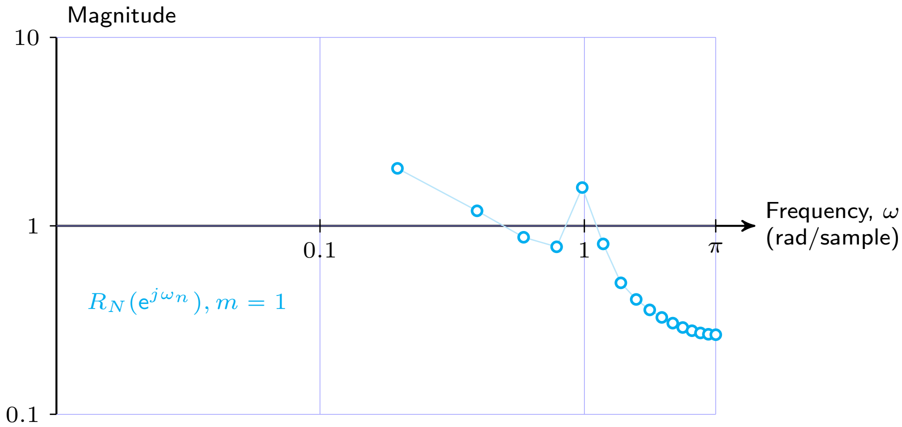
Periodic signal with period $M = 32$.

$r(k)$ for $k = 0, \ldots, N-1$. $(N = mM)$



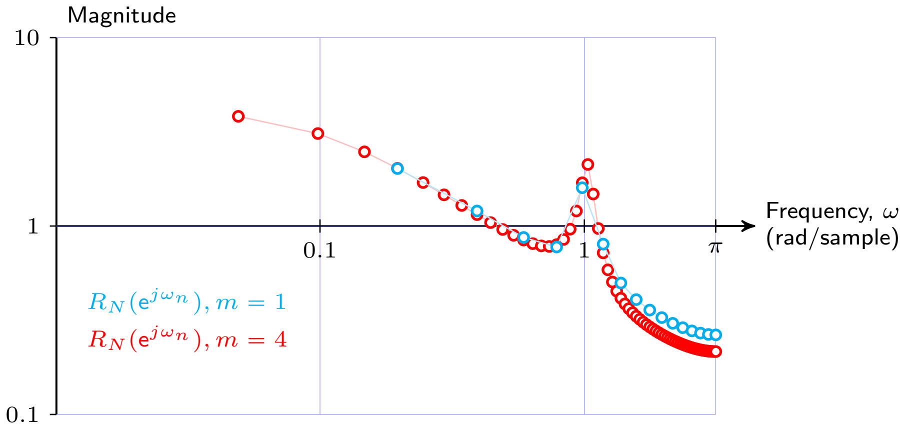
Transient responses

$R_N(\mathrm{e}^{j\omega_n})$ for $n = 0, \ldots, N-1$. $\qquad (N = mM)$



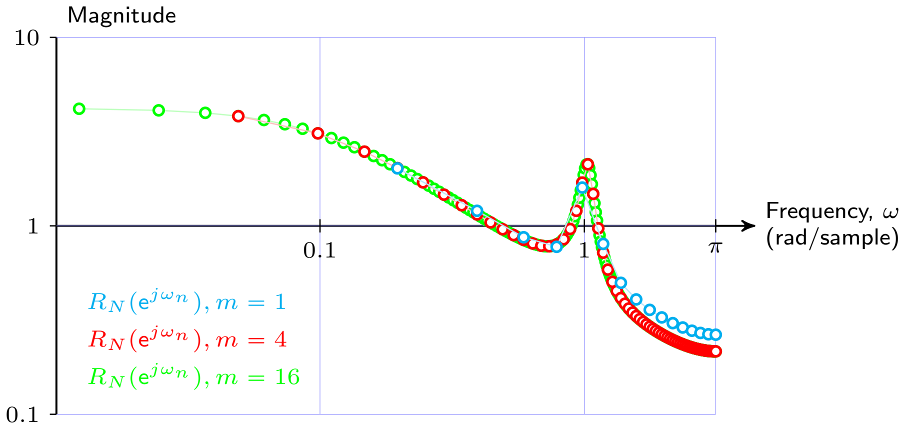
Transient responses

$R_N(\mathrm{e}^{j\omega_n})$ for $n = 0, \ldots, N-1$. $\qquad (N = mM)$



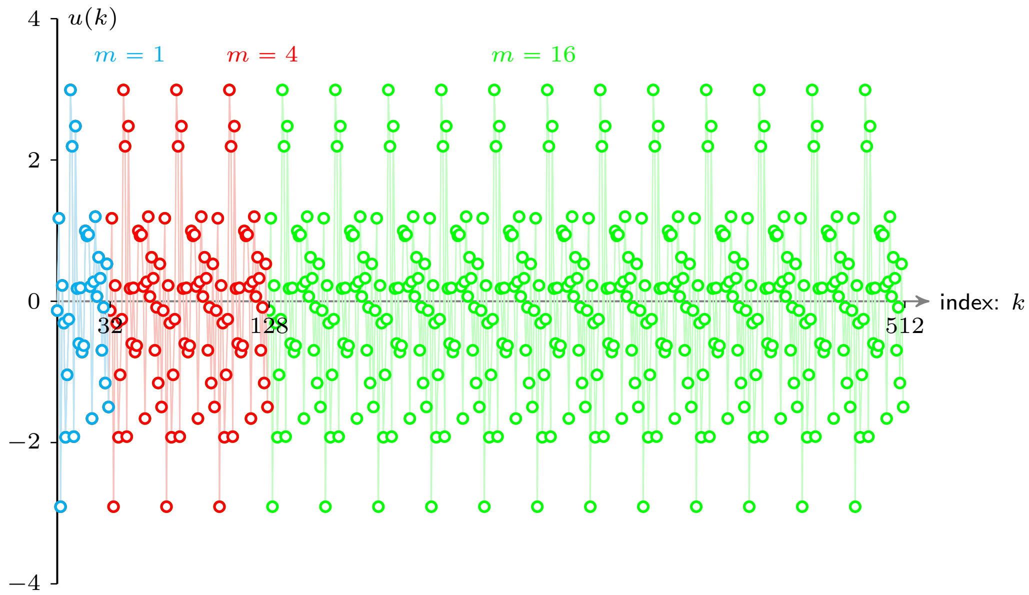
Transient responses

$R_N(\mathrm{e}^{j\omega_n})$ for $n = 0, \ldots, N-1$. $\qquad (N = mM)$



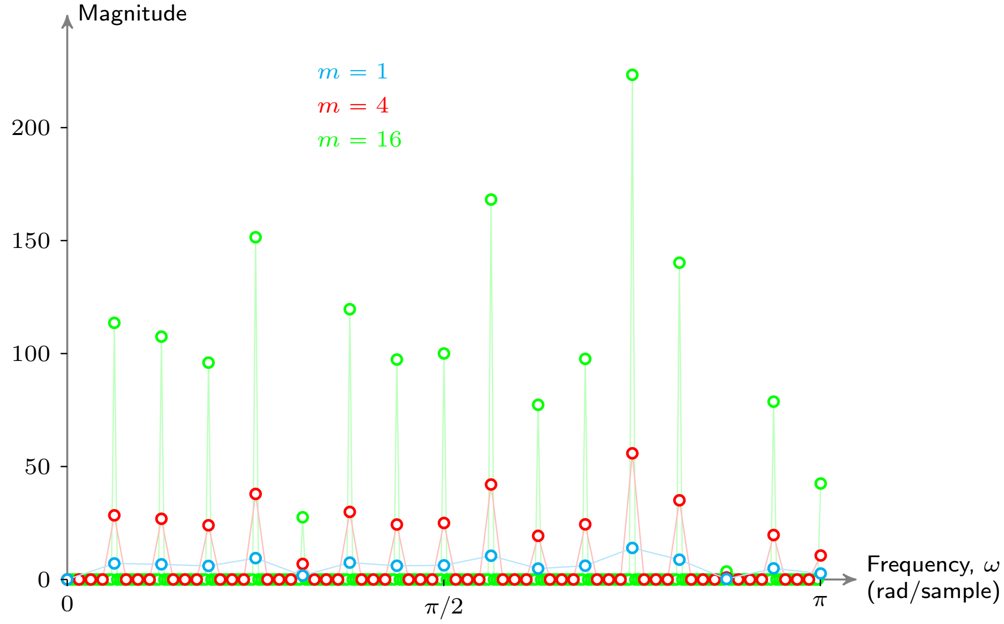
Transient responses: input excitation

$u(k)$ for $k = 0, \ldots, N-1$. where $(N = mM, \quad M = 32$ and $m = 1, 4, 16)$.



Transient responses: input excitation

$U_N(\mathrm{e}^{j\omega_n})$ for $n = 0, \ldots, N/2$ where $N = mM$, $M = 32$ and $m = 1, 4, 16$.



ETFE Transient error properties

ETFE transient bias error

$$\hat{G}(\mathrm{e}^{j\omega_n}) = \frac{Y_N(\mathrm{e}^{j\omega_n})}{U_N(\mathrm{e}^{j\omega_n})} = G_0(\mathrm{e}^{j\omega_n}) + \frac{R_N(\mathrm{e}^{j\omega_n})}{U_N(\mathrm{e}^{j\omega_n})} + \frac{V_N(\mathrm{e}^{j\omega_n})}{U_N(\mathrm{e}^{j\omega_n})}$$

Periodic $u(k)$

As $N = mM$, $m \longrightarrow \infty$,

$$\left|U_N(\mathrm{e}^{j\omega_n})\right|^2 = m^2 \left|U_M(\mathrm{e}^{j\omega_n})\right|^2$$

Random $u(k)$

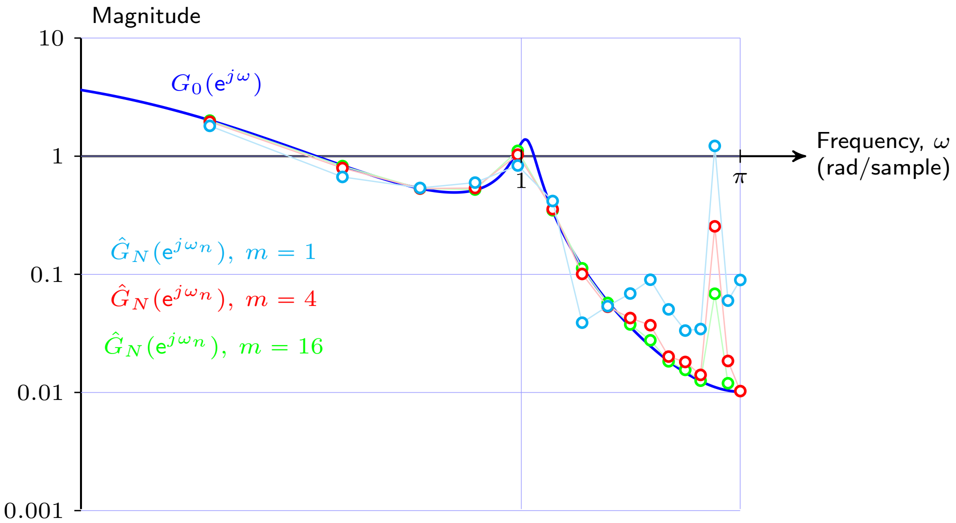
As $N \longrightarrow \infty$,

$$E\left\{\left|U_N(\mathrm{e}^{j\omega_n})\right|^2\right\} \longrightarrow N\,\phi_u(\mathrm{e}^{j\omega_n})$$

So $\left|\dfrac{R_N(\mathrm{e}^{j\omega_n})}{U_N(\mathrm{e}^{j\omega_n})}\right|^2 \longrightarrow 0$ with rate $\begin{cases} 1/N^2 & \text{for periodic inputs; or} \\ 1/N & \text{for random inputs} \end{cases}$

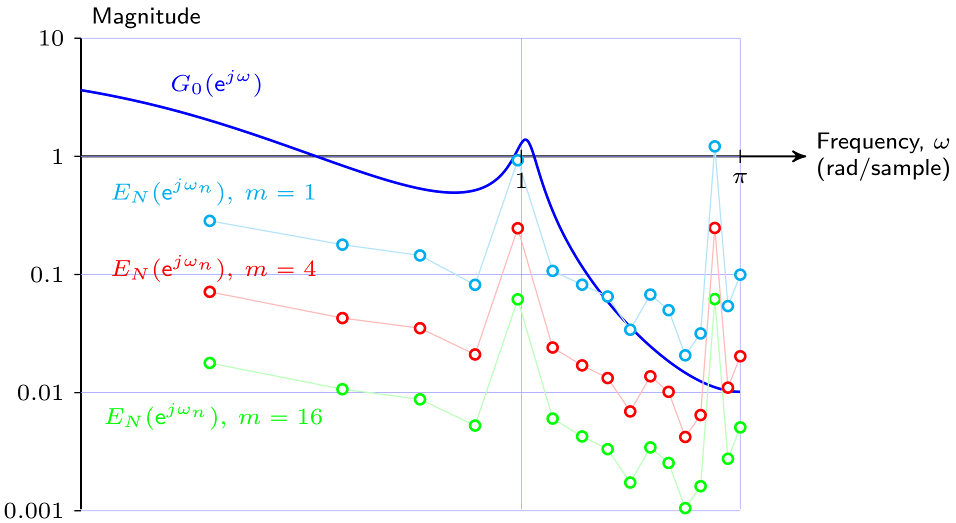
Transient response example

Estimates: $\hat{G}_N(\mathrm{e}^{j\omega_n}) = Y_N(\mathrm{e}^{j\omega_n})/U_N(\mathrm{e}^{j\omega_n})$ for increasing m.



Transient response example

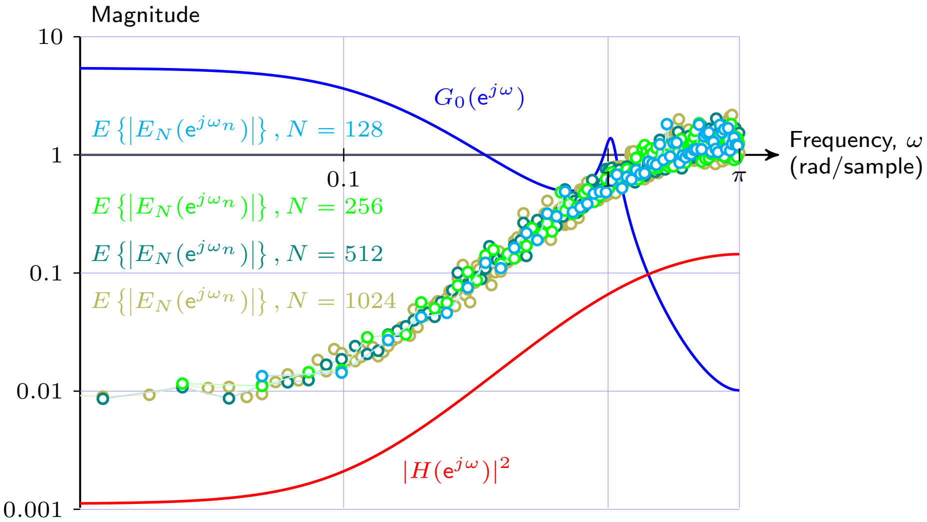
Transient error decay: $E_N(\mathrm{e}^{j\omega_n}) = \hat{G}_N(\mathrm{e}^{j\omega_n}) - G_0(\mathrm{e}^{j\omega_n})$



ETFE example: average square error with noisy data

Previous example with noise (no transient): period $M = N$ in each experiment.

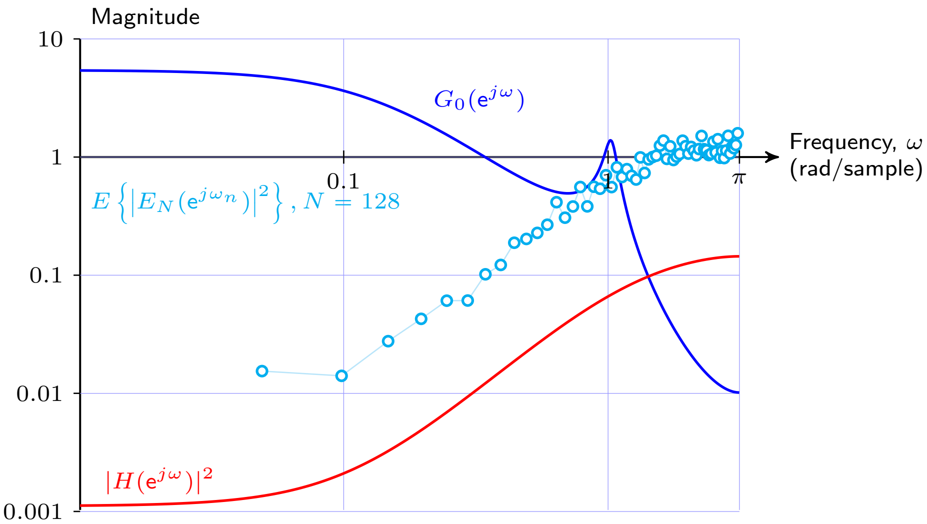
$$E\left\{\left|E_N(\mathrm{e}^{j\omega_n})\right|^2\right\} \approx \frac{1}{10,000}\sum_{l=1}^{10,000}\left|G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})\right|^2$$



ETFE example: average square error with periodic signals

Noisy example (no transient): period $M = 128$. Larger $N \implies$ more periods.

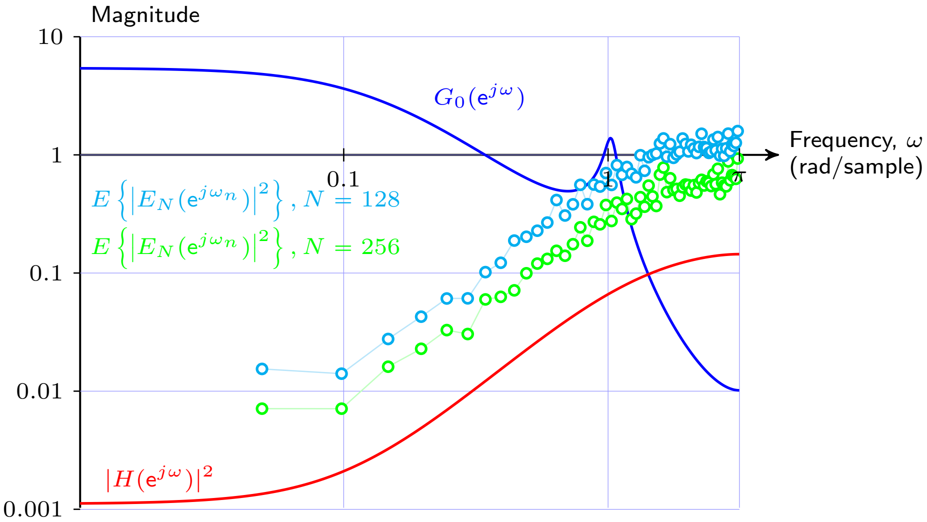
$$E\left\{\left|E_N(\mathrm{e}^{j\omega_n})\right|^2\right\} \approx \frac{1}{10,000}\sum_{l=1}^{10,000}\left|G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})\right|^2$$



ETFE example: average square error with periodic signals

Noisy example (no transient): period $M = 128$. Larger $N \implies$ more periods.

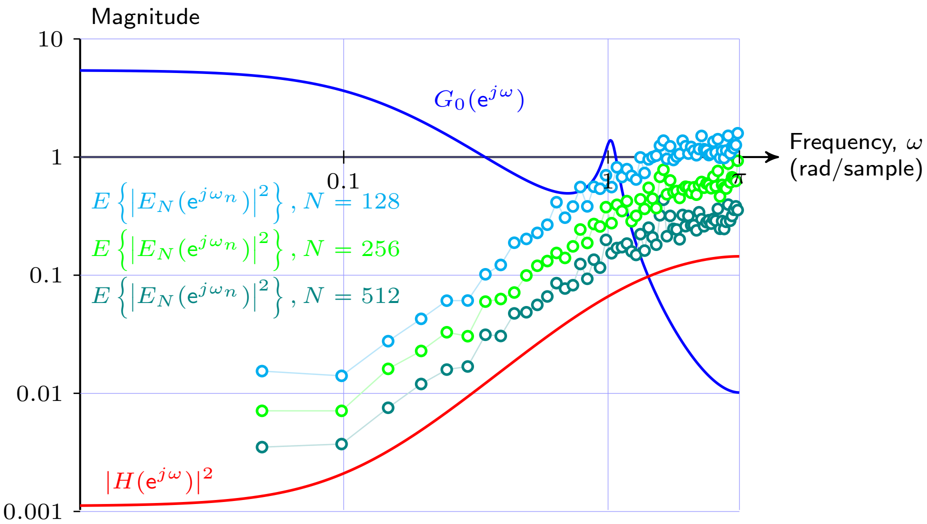
$$E\left\{\left|E_N(\mathrm{e}^{j\omega_n})\right|^2\right\} \approx \frac{1}{10,000}\sum_{l=1}^{10,000}\left|G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})\right|^2$$



ETFE example: average square error with periodic signals

Noisy example (no transient): period $M = 128$. Larger $N \implies$ more periods.

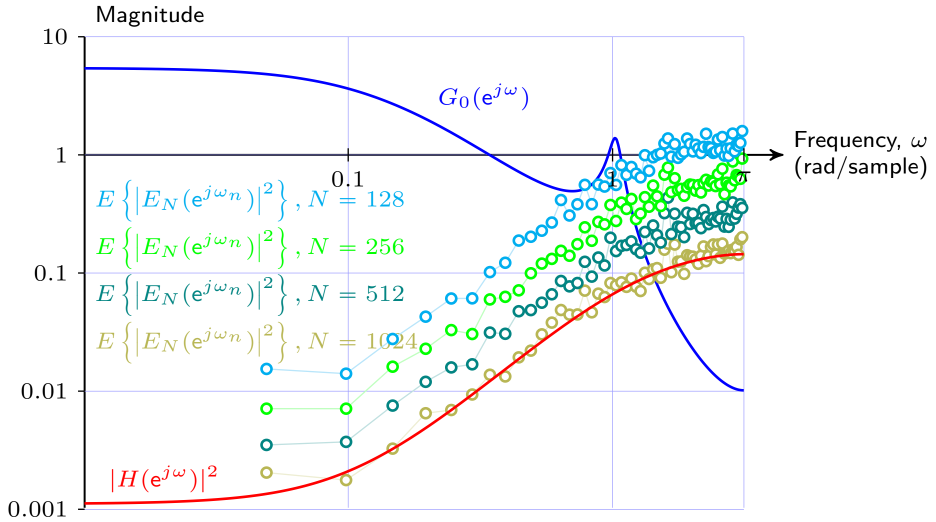
$$E\left\{\left|E_N(\mathrm{e}^{j\omega_n})\right|^2\right\} \approx \frac{1}{10,000}\sum_{l=1}^{10,000}\left|G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})\right|^2$$



ETFE example: average square error with periodic signals

Noisy example (no transient): period $M = 128$. Larger $N \implies$ more periods.

$$E\left\{\left|E_N(\mathrm{e}^{j\omega_n})\right|^2\right\} \approx \frac{1}{10,000}\sum_{l=1}^{10,000}\left|G_0(\mathrm{e}^{j\omega_n}) - \hat{G}_N(\mathrm{e}^{j\omega_n})\right|^2$$



Bibliography

Fourier transforms:
A.V. Oppenheim, A.S. Wilsky & S.H. Nawab, *Signals & Systems*, Prentice-Hall, 2nd Ed., 1996.

Empirical Transfer Function Estimation (ETFE):
Lennart Ljung, *System Identification; Theory for the User,* (see Section 6.3) Prentice-Hall, 2nd Ed., 1999.

Spectral estimation:
Lennart Ljung, *System Identification; Theory for the User,* (see Section 6.4) Prentice-Hall, 2nd Ed., 1999.

P. Stoica & R. Moses, *Introduction to Spectral Analysis* (see Chapters 1 and 2), Prentice-Hall, 1997.