Control Systems 2

Lecture 8: MIMO stability and stabilisation

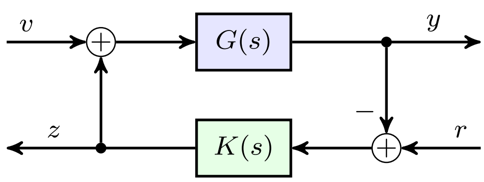
Roy Smith

8:15, Wednesday 13th April, 2021

Internal stability

Definition

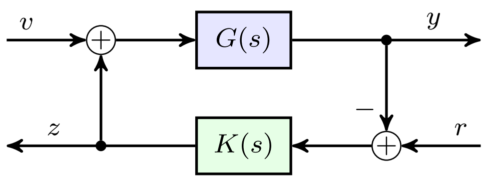
A system is internally stable if for all initial conditions, and all bounded signals injected at any place in the system, all states remain bounded for all future time.



$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

Are all four transfer functions stable?

Internal stability



Closed-loop input-output relationship:

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

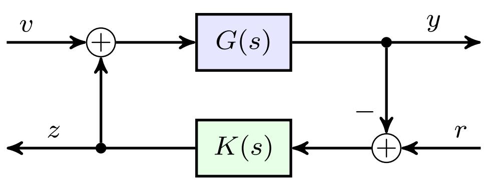
Are all four transfer functions stable?

Example:

$$G(s) = \frac{s-1}{s+1} \qquad K(s) = \frac{k(s+1)}{s(s-1)}$$

Work out $S(s)$, $T(s)$, and all four transfer functions, $N_{ij}(s)$.

Internal stability



Example:

$$G(s) = \frac{s-1}{s+1} \qquad K(s) = \frac{k(s+1)}{s(s-1)}$$

MIMO concepts: transfer function matrices

$$y(s) = \begin{bmatrix} y_1(s) \\ \vdots \\ y_{n_y}(s) \end{bmatrix} = G(s)u(s) = \begin{bmatrix} G_{11}(s) & \dots & G_{1n_u}(s) \\ \vdots & & \vdots \\ G_{n_y1}(s) & \dots & G_{n_yn_u}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ \vdots \\ u_{n_u}(s) \end{bmatrix}$$

$$G(s) = \begin{bmatrix} G_{11}(s) & \dots & G_{1n_u}(s) \\ \vdots & & \vdots \\ G_{n_y1}(s) & \dots & G_{n_yn_u}(s) \end{bmatrix} = \begin{bmatrix} \frac{b_{11}(s)}{a_{11}(s)} & \dots & \frac{b_{1n_u}(s)}{a_{1n_u}(s)} \\ \vdots & & \vdots \\ \frac{b_{n_y1}(s)}{a_{n_y1}(s)} & \dots & \frac{b_{n_yn_u}(s)}{a_{n_yn_u}(s)} \end{bmatrix}$$

$$= C(sI - A)^{-1}B + D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

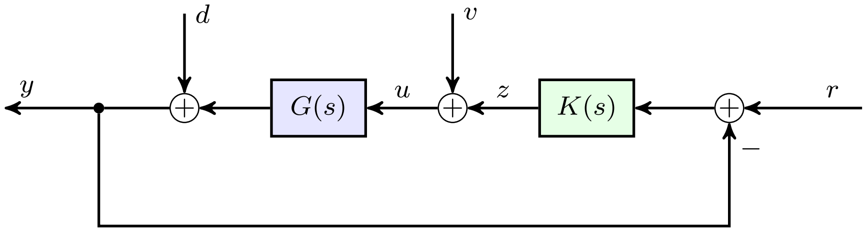
with $A \in \mathcal{R}^{n \times n}, B \in \mathcal{R}^{n \times n_u}, C \in \mathcal{R}^{n_y \times n}, D \in \mathcal{R}^{n_y \times n_u}$.

MIMO block diagrams

Non-commutative

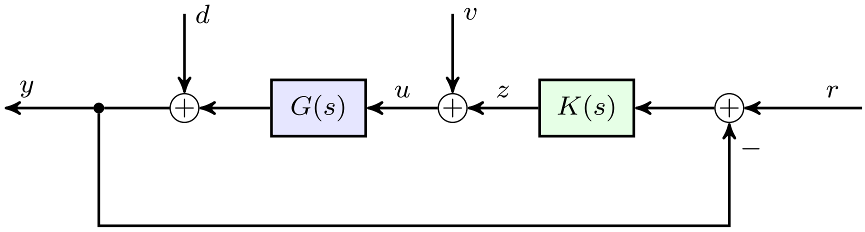
MIMO block diagrams

"Push-through" rule



$$GK(I+GK)^{-1} = G(I+KG)^{-1}K = (I+GK)^{-1}GK$$

MIMO sensitivity and complementary sensitivity functions

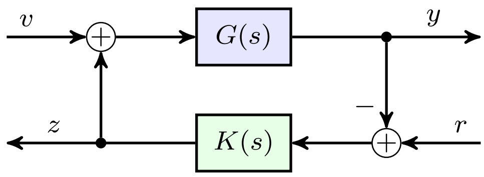


$$y = \underbrace{(I+GK)^{-1}GK}_{T_O}\ r + (I+GK)^{-1}Gv \ + \underbrace{(I+GK)^{-1}}_{S_O}\ d$$

$$u = (I+KG)^{-1}Kr \ + \underbrace{(I+KG)^{-1}}_{S_I}\ v - (I+KG)^{-1}Kd$$

$$z = (I+KG)^{-1}Kr \ - \underbrace{(I+KG)^{-1}KG}_{T_I}\ v - (I+KG)^{-1}Kd$$

Internal stability



$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix} = \begin{bmatrix} S_{\mathrm{O}}(s)G(s) & T_{\mathrm{O}}(s) \\ -T_{\mathrm{I}}(s) & S_{\mathrm{I}}(s)K(s) \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$(S_{\mathrm{O}} = (I + GK)^{-1} \in \mathcal{C}^{n_y \times n_y},\ S_{\mathrm{I}} = (I + KG)^{-1} \in \mathcal{C}^{n_u \times n_u})$

Internally stable $\iff$ $T_{\mathrm{O}}(s), S_{\mathrm{O}}(s)G(s)$ and $K(s)S_{\mathrm{O}}(s)$ stable.

Or, equivalently,:

Internally stable $\iff$ $S_{\mathrm{O}}(s)$ stable and no RHP cancellations in $G(s)K(s)$. (minimal realisations of GK & KG contain all RHP poles).

Internal stability

Consequences:

If $G(s)$ has a RHP-zero at z then (if internally stable),

$$\left.\begin{array}{l} L_{\mathrm{O}}(s) = G(s)K(s) \\ T_{\mathrm{O}}(s) = G(s)K(s)(I - G(s)K(s))^{-1} \\ S_{\mathrm{O}}(s)G(s) = (I + G(s)K(s))^{-1}G(s) \\ L_{\mathrm{I}}(s) = K(s)G(s)) \\ T_{\mathrm{I}}(s) = K(s)G(s)(I + K(s)G(s))^{-1} \end{array}\right\} \text{have a RHP-zero at } z.$$

Feedback will not move (or remove) the RHP-zero from the closed-loop transfer functions.

Internal stability

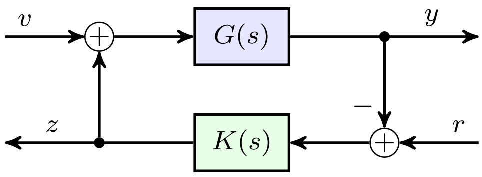
Consequences:

If $G(s)$ has a RHP-pole at p then (if internally stable),

$$\left.\begin{array}{l} L_O(s) = G(s)K(s) \\ L_I(s) = K(s)G(s)) \end{array}\right\} \text{have a RHP-pole at } p,$$

$$\left.\begin{array}{l} S_O(s) = (I + G(s)K(s))^{-1} \\ K(s)S_O(s) = K(s)(I + G(s)K(s))^{-1} \\ S_I(s) = (I + K(s)G(s))^{-1} \end{array}\right\} \text{have a RHP-zero at } p.$$

Stabilising controllers



$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} S_O G & T_O \\ -T_I & S_I K \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

Stable plant case:

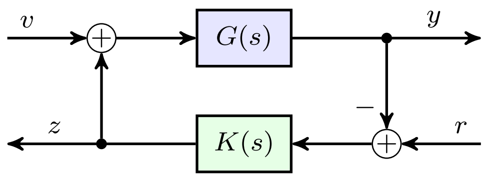
Define:

$$Q(s) = K(s)(I + G(s)K(s))^{-1}$$

Then,

$$\left.\begin{array}{l} S_O G = (I + GK)^{-1}G = (I - GQ)G \\ T_O = GK(I + GK)^{-1} = GQ \\ T_I = KG(I + KG)^{-1} = QG \\ S_I K = (I + KG)^{-1}K = Q \end{array}\right\} \text{are stable if } Q \text{ is stable.}$$

Stabilising controllers



$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} S_O G & T_O \\ -T_I & S_I K \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

Stable plant case:

The converse is true:

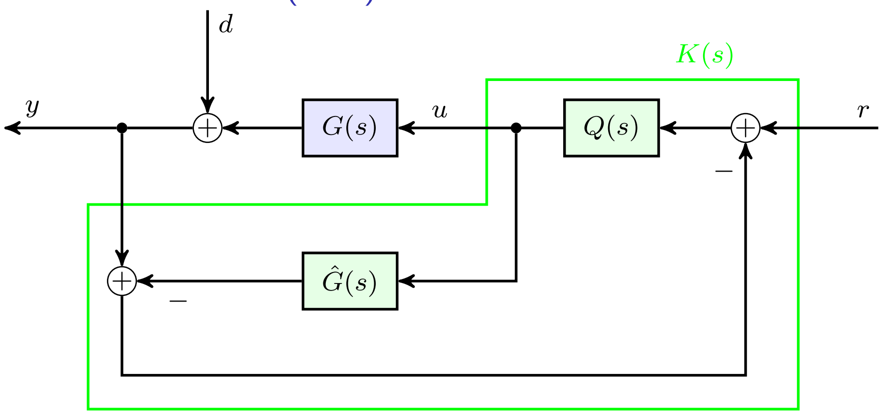
For every stabilising controller $K(s)$,

$$Q(s) = K(s)(I + G(s)K(s))^{-1}, \qquad \text{is also stable.}$$

This is a parameterisation of all stabilising controllers.

Q-parameterisation or Youla parametrisation.

Internal model control (IMC)



Assume that $G(s)$ is stable and a perfect model: $G(s) = \hat{G}(s)$

$$y = d + Gu = \underbrace{GQ}_{T_O}\ r + \underbrace{(I - GQ)}_{S_O}\ d$$

$$u = \begin{bmatrix} (I - QG)^{-1}Q & -(I - QG)^{-1}Q \end{bmatrix} \begin{bmatrix} r \\ y \end{bmatrix} = \begin{bmatrix} K & -K \end{bmatrix} \begin{bmatrix} r \\ y \end{bmatrix} = K(r - y)$$

IMC design (for stable $G(s)$)

$$Q = K(I + GK)^{-1}, \qquad K = (I - QG)^{-1}Q$$

Closed-loop in linear in Q:

$$T(s) = G(s)Q(s)$$

Design approach:

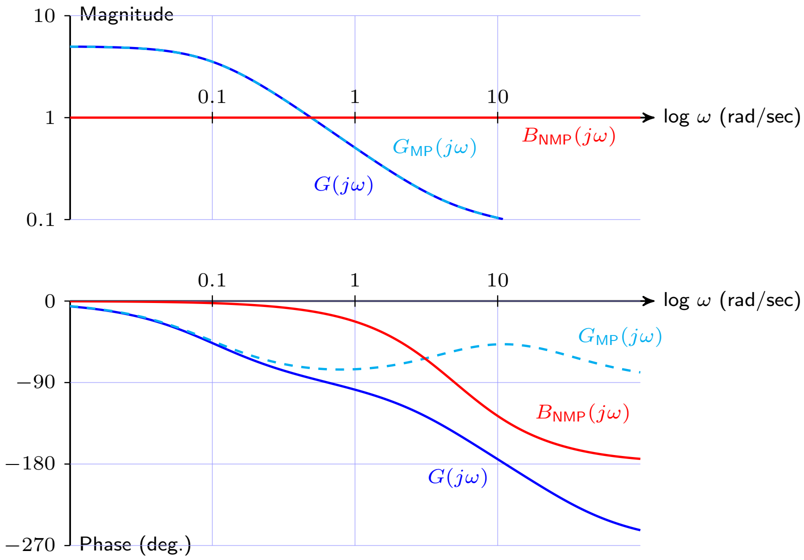
$$Q(s) = G(s)^{-1}T_{\text{ideal}}(s)$$

or if $G(s) = B_{\text{NMP}}(s)G_{\text{MP}}(s), \quad Q(s) = G_{\text{MP}}(s)^{-1}T_{\text{ideal}}(s).$

- Cannot invert non-minimum phase parts of $G(s)$.
- Relative degree of $T_{\text{ideal}}(s) \geq$ relative degree of $G_{\text{MP}}(s)$ makes $Q(s)$ proper.

IMC design example

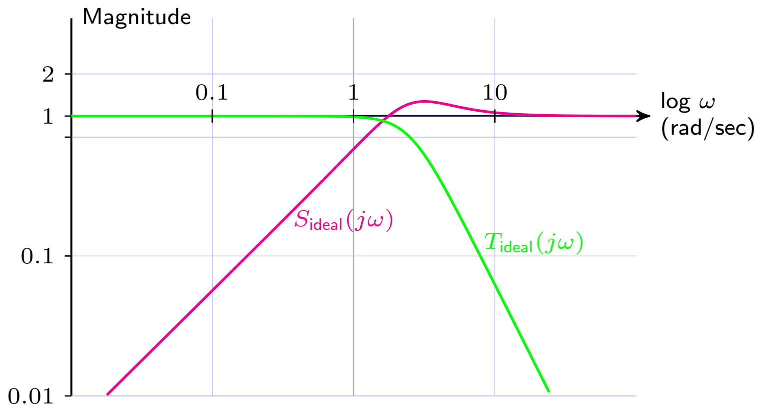
$$G(s) = \frac{5}{(1+5s)}\frac{(1-s/5)}{(1+s/25)} = \underbrace{\frac{(1-s/5)}{(1+s/5)}}_{B_{\text{NMP}}(s)} \ \underbrace{\frac{5}{(1+5s)}\frac{(1+s/5)}{(1+s/25)}}_{G_{\text{MP}}(s)}$$



IMC design example

Select a desired closed-loop transfer function:

$$T_{\text{ideal}}(s) = \frac{\omega_c^2}{(s^2 + \sqrt{2}\omega_c s + \omega_c^2)}, \quad \omega_c = 2.5, \qquad S_{\text{ideal}}(s) = 1 - T_{\text{ideal}}(s).$$



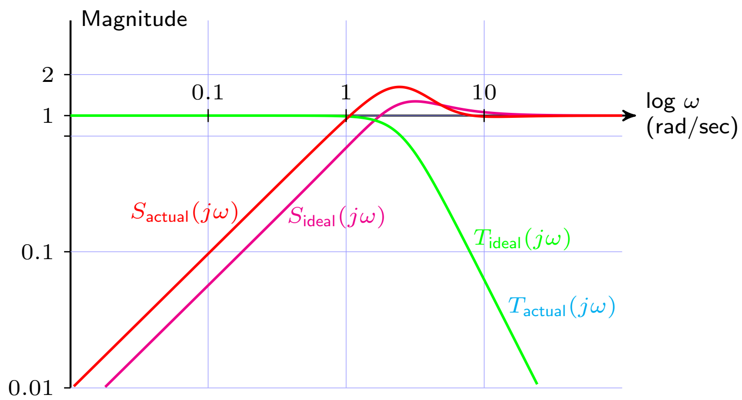
IMC design example

Invert $G_{\text{MP}}(s)$ to get $Q(s)$.

$$Q(s) = G_{\text{MP}}(s)^{-1} T_{\text{ideal}}(s) = \frac{(1+5s)}{5} \frac{(1+s/25)}{(1+s/5)} \frac{\omega_c^2}{(s^2 + \sqrt{2}\omega_c s + \omega_c^2)}$$

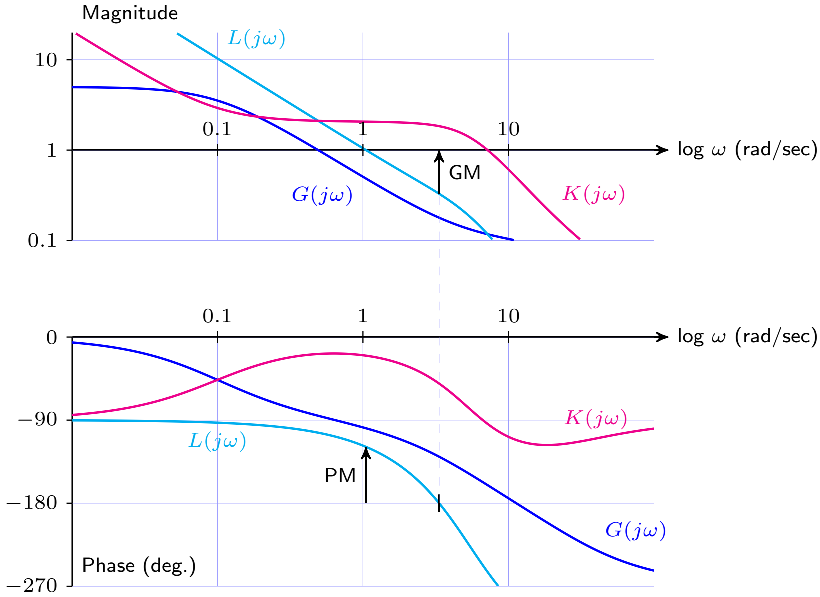
The actual closed-loop, $T(s)$, is:

$$T(s) = G(s)Q(s) = B_{\text{NMP}}(s) T_{\text{ideal}}(s) = \frac{(1-s/5)}{(1+s/5)} \frac{\omega_c^2}{(s^2 + \sqrt{2}\omega_c s + \omega_c^2)}.$$



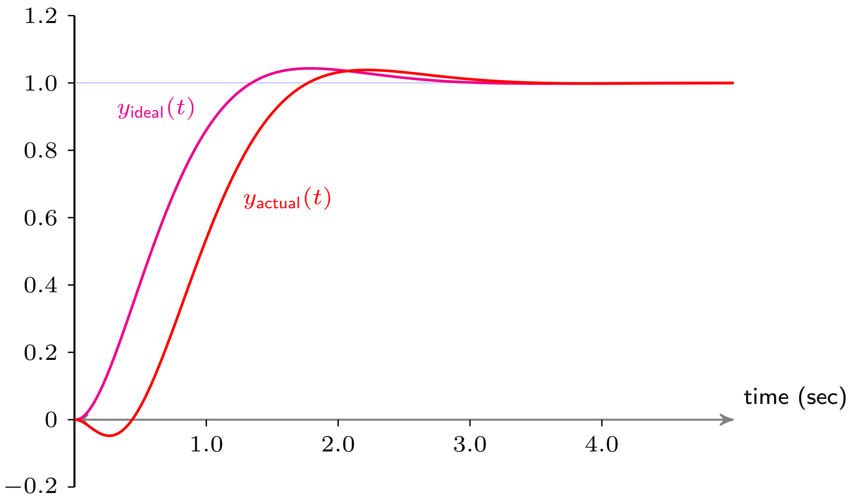
IMC design example

Controller: $K(s) = (I - Q(s)G(s))^{-1}Q(s)$ (5th order controller)

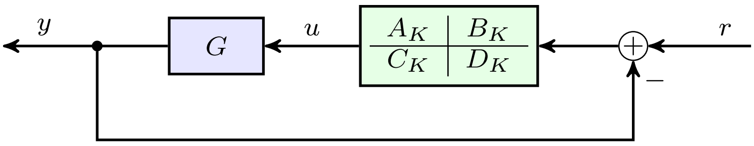


IMC design example

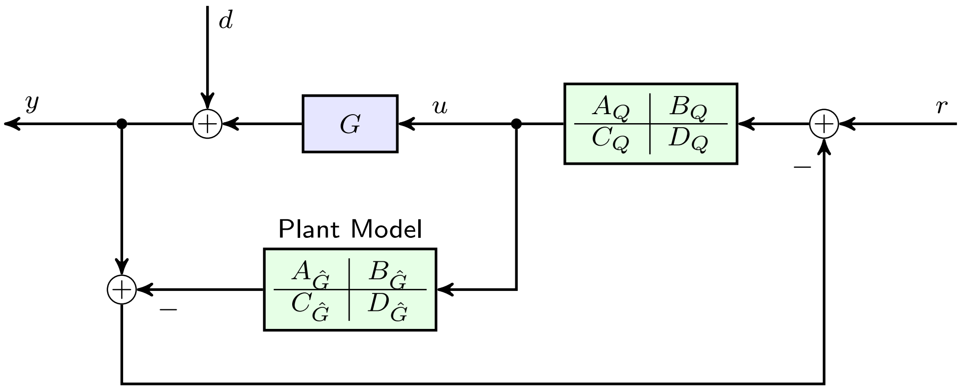
Closed-loop unit step responses:



IMC implementation

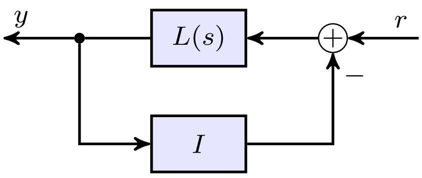


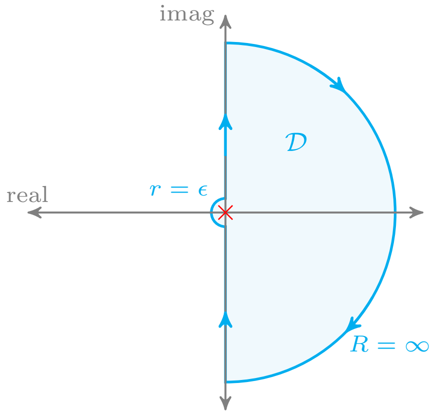
Or ...



MIMO Nyquist stability analysis

For a minimal $L(s)$,





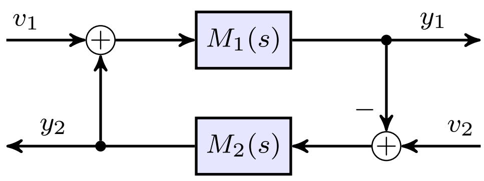
Closed-loop exponential stability

If and only if,

i) $\det(I + L(s)) \neq 0$, for all $s \in \mathcal{D}$

ii) The number of CCW encirclements of the origin by $\det(I + L(s))$, as s traverses the boundary of $\mathcal{D}$, is equal to the number of unstable poles in $L(s)$.

Small gain theorem



A sufficient condition for stability

Given $M_1(s)$ and $M_2(s)$ stable and minimal with,

$$\|M_1(s)\| = \gamma_1 \quad \text{and} \quad \|M_2(s)\| = \gamma_2$$

If $\gamma_1\gamma_2 < 1$ then
the closed-loop interconnection is stable.

This holds for any induced norm (with the same norm for input and output signals).

$\mathcal{H}_\infty$ norm

The $\mathcal{H}_\infty$ norm is a measure of the "size" or "gain" of a system.

If $y(s) = G(s)u(s)$ (and stable) then,

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{u(j\omega)\neq 0} \frac{\|y(j\omega)\|_2}{\|u(j\omega)\|_2} \qquad \text{(induced norm with the space)}$$

$$= \sup_{u(s)\neq 0} \frac{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} y(j\omega)^T y(j\omega) d\omega\right)^{1/2}}{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} u(j\omega)^T u(j\omega) d\omega\right)^{1/2}}$$

$$= \max_{\omega} \overline{\sigma}\left(G(j\omega)\right) \quad = \|G(s)\|_\infty \quad \text{(alternative notation)}$$

$\mathcal{H}_\infty$ is the set of stable, $\mathcal{H}_\infty$-norm bounded transfer functions.

$\mathcal{H}_2$ norm

Another measure of the "size" or "gain" of a system.

$$\|G(s)\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{trace}\left(G(j\omega)^* G(j\omega)\right) \, d\omega \right)^{1/2}$$

The integrand is the Frobenius norm squared of the frequency response:

$$\operatorname{trace}\left(G(j\omega)^* G(j\omega)\right) = \sum_{i,j} |G_{ij}(j\omega)|^2 = \|G(j\omega)\|_F^2.$$

Via Parseval's theorem:

$$\|G(s)\|_{\mathcal{H}_2} = \|g(t)\|_{\mathcal{H}_2} = \left(\int_0^{\infty} \operatorname{trace}\left(g(\tau)^T g(\tau) \right) \, d\tau \right)^{1/2}$$

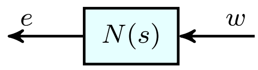
$\mathcal{H}_2$ norm

For state-space representations:

$$\begin{aligned} \|G(s)\|_{\mathcal{H}_2}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{trace}\left(G(j\omega)^* G(j\omega)\right) \, d\omega \\ &= \int_0^{\infty} \operatorname{trace}\left(B^T e^{A^T \tau} C^T C e^{A\tau} B \right) \, d\tau \\ &= \operatorname{trace}(B^T W_o B) \qquad (W_o: \text{ observability Grammian}) \\ &= \operatorname{trace}(C W_c C^T) \qquad (W_c: \text{ controllability Grammian}) \end{aligned}$$

(writing $\|G(s)\|_{\mathcal{H}_2}^2$ avoids square roots)

Nominal performance norm tests



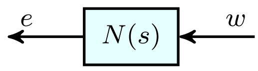
$\|N(s)\|_{\mathcal{H}_2} < 1$ implies:

- If $w(t) = \delta(t)$, then $\|e(t)\|_2 < 1$.
- If $\|w(t)\|_2 < 1$, then $\max\limits_t |e(t)| < 1$.
- If $w(t)$ is unit variance white noise, the $\text{var}(e(t)) < 1$.

$\|N(s)\|_{\mathcal{H}_\infty} < 1$ implies:

- If $w(t) = \sin(\omega t)$ then, $\max\limits_t |e(t)| < 1$.
- If $\|w(t)\|_2 < 1$ then, $\|e(t)\|_2 < 1$.

System norm comparison



$\mathcal{H}_2$ norm

- Useful nominal performance measure.
- Linear quadratic (LQ) design methods use this norm.
- Minimises "square errors".

$\mathcal{H}_\infty$ norm

- Useful nominal performance measure.
- Minimises "worst-case" errors.
- Induced norm: small-gain applies.
- Very useful for robustness analysis.

Notes and references

Skogestad & Postlethwaite (2nd Ed.)

Internal stability: section 4.7

Stabilising controllers: section 4.8

Stability analysis: section 4.9

System norms: section 4.10

IMC design: section 2.7

IMC design

Robust Process Control, Manfred Morari & Evanghelos Zafiriou, Prentice-Hall, 1989. (Chapters 3–6).

MIMO Nyquist criterion

"On the Generalized Nyquist Stability Criterion," C.A. Desoer and Y.-T. Wang, *IEEE Trans. Auto. Control*, v. 25, no. 2, pp. 187–196, 1980.