Control Systems 2

Lecture 7: System theory: controllability, observability, stability, poles and zeros

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State-space representations

Idea:

- Transfer function is a ratio of polynomials (say n^{th} order).
- Reformulate n^{th} order differential equation as a first order matrix differential equation (with matrix dimension n).

Motivation:

- Design & analysis methods use linear algebra (c.f. polynomial algebra).
- Easy to handle large systems (using MATLAB).
- Easy to handle systems with multiple inputs and outputs.
- Easy to simulate systems.
- Numerically better than polynomial based calculations.

State-space representations

System descriptions (linear, time-invariant systems)

State vector: $x(t) \in \mathcal{R}^n$ (evolving with time)

Input vector: $u(t) \in \mathcal{R}^{n_u}$

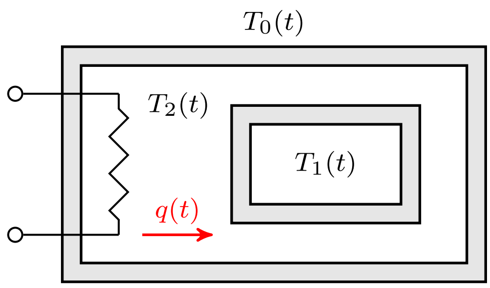
Output vector: $y(t) \in \mathcal{R}^{n_y}$

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad A \in \mathcal{R}^{n \times n}$$
$$y(t) = Cx(t) + Du(t). \quad D \in \mathcal{R}^{n_y \times n_u}$$

or:

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

Thermal control system example



$T_i(t)$	temperature of volume i
$q(t)$	heat flux into the volume 2
m_i	mass of volume i
c_i	specific heat for volume i
k_{ij}	thermal conductance for i, j interface

Volume 1: $$m_1 c_1 \frac{dT_1(t)}{dt} = k_{12}(T_2(t) - T_1(t))$$

Volume 2: $$m_2 c_2 \frac{dT_2(t)}{dt} = -k_{12}(T_2(t) - T_1(t)) - k_{20}(T_2(t) - T_0(t)) + q(t)$$

Thermal control system example

Output: $T_1(t)$; Inputs: $q(t)$ and $T_0(t)$; State: $x(t) = \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix}$.

$$\frac{dT_1(t)}{dt} = \frac{-k_{12}}{m_1 c_1} T_1(t) + \frac{k_{12}}{m_1 c_1} T_2(t)$$

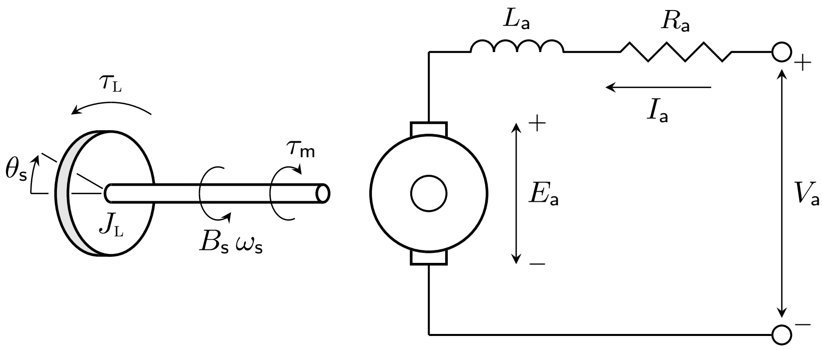
$$\frac{dT_2(t)}{dt} = \frac{k_{12}}{m_2 c_2} T_1(t) + \frac{-k_{12} - k_{20}}{m_2 c_2} T_2(t) + \frac{k_{20}}{m_2 c_2} T_0(t) + \frac{1}{m_2 c_2} q(t)$$

So,

$$\begin{bmatrix} \frac{dT_1(t)}{dt} \\ \frac{dT_2(t)}{dt} \end{bmatrix} = A \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix} + B \begin{bmatrix} T_0(t) \\ q(t) \end{bmatrix}, \quad A = \begin{bmatrix} \frac{-k_{12}}{m_1 c_1} & \frac{k_{12}}{m_1 c_1} \\ \frac{k_{12}}{m_2 c_2} & \frac{-k_{12} - k_{20}}{m_2 c_2} \end{bmatrix}$$

Exercise: find B, C and D.

Example: DC motor with rotational load



"Back EMF": $E_a = K \frac{d\theta_s}{dt}$

Kirchoff's voltage law: $V_a = E_a + L_a \frac{dI_a}{dt} + R_a I_a$

Motor torque: $\tau_m = K_\tau I_a$

Friction torque: $\tau_f = B_s\,\omega_s = B_s \frac{d\theta_s}{dt}$

Torque balance: $J_L \frac{d^2\theta_s}{dt^2} + \tau_f = \tau_m$

State-space representations

Solution (zero input case: $u(t) = 0$)

$$\frac{dx(t)}{dt} = A\,x(t).$$

Unilateral Laplace Transform:

$$sx(s) - x(0) = A\,x(s) \quad \Longrightarrow \quad x(s) = (sI - A)^{-1}x(0).$$

Taking an inverse Laplace transform gives,

$$x(t) \quad = \quad \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} x(0) \quad = \quad \Phi(t)x(0).$$

$\Phi(t)$, is also known as the *"State Transition Matrix"*.

State-space representations

Solution (forced input case: $u(t) \neq 0$)

$$\frac{dx(t)}{dt} = A\,x(t) \;+\; B\,u(t).$$

Taking Laplace transforms gives,

$$sx(s) - x(0) = A\,x(s) \;+\; B\,u(s),$$

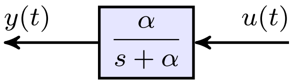
$$x(s) = (sI - A)^{-1}\,x(0) \;+\; (sI - A)^{-1}\,B\,u(s).$$

Now the inverse Laplace gives,

$$x(t) \quad = \quad \underbrace{\Phi(t)\,x(0)}_{\text{zero-input solution}} \quad + \quad \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau.}_{\text{convolution of } \Phi(t) \text{ and } B\,u(t)}$$

State-space example

First order system ($\alpha > 0$)



Take the initial conditions to be zero. The differential equation is,

$$\frac{dy(t)}{dt} + \alpha\, y(t) = \alpha\, u(t).$$

Define the state as, $x(t) = y(t)$, then,

$$\frac{dx(t)}{dt} = -\alpha\, x(t) + \alpha u(t), \qquad y(t) = x(t) \quad \Longrightarrow \quad \begin{matrix} A = -\alpha & B = \alpha \\ C = 1 & D = 0 \end{matrix}$$

State-space example

State transition matrix

$$\Phi(t) = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+\alpha}\right\} = \mathrm{e}^{-\alpha t} \quad \text{(impulse reponse)}$$

$$x(t) = \mathrm{e}^{-\alpha t}\, x(0) + \int_0^t \mathrm{e}^{-\alpha(t-\tau)}\, \alpha\, u(\tau)\, d\tau.$$

Step response:

Initial condition: $x(0) = 0$, Input: $u(t) = 1, \quad t \geq 0$.

$$y(t) = x(t) = \mathrm{e}^{-\alpha t}\, 0 + \int_0^t \mathrm{e}^{-\alpha(t-\tau)}\, \alpha\, d\tau = \mathrm{e}^{-\alpha t} \int_0^t \mathrm{e}^{\alpha\tau}\, \alpha\, d\tau,$$

$$= \mathrm{e}^{-\alpha t}\left[\mathrm{e}^{\alpha\tau}\,|_{\tau=t} - \mathrm{e}^{\alpha\tau}\,|_{\tau=0}\right] = \mathrm{e}^{-\alpha t}\left(\mathrm{e}^{\alpha t} - 1\right)$$

$$= 1 - \mathrm{e}^{-\alpha t}.$$

Matrix exponential

State transition matrix

The zero-input solution (via repeated differentiation) is:

$$x(t) \quad = \quad x(0) + At\,x(0) \;+\; \tfrac{A^2}{2}t^2\,x(0) \;+\; \tfrac{A^3}{3!}t^3\,x(0) \;+\cdots$$

Matrix exponential

State transition matrix

The zero-input solution (via repeated differentiation) is:

$$\begin{aligned} x(t) \quad &= \quad x(0) + At\,x(0) \;+\; \tfrac{A^2}{2}t^2\,x(0) \;+\; \tfrac{A^3}{3!}t^3\,x(0) \;+\cdots \\ &= \quad \underbrace{\left[I + At \;+\; \frac{A^2}{2}t^2 \;+\; \frac{A^3}{3!}t^3 \;+\cdots \right]}_{\text{define this as } \mathrm{e}^{At}} \;\; x(0). \end{aligned}$$

$$x(t) \;=\; \mathrm{e}^{At}\,x(0) \qquad \text{and so} \qquad \Phi(t) \;=\; \mathrm{e}^{At}.$$

Matrix exponential

State transition matrix

The zero-input solution (via repeated differentiation) is:

$$x(t) = x(0) + At\,x(0) + \tfrac{A^2}{2}t^2\,x(0) + \tfrac{A^3}{3!}t^3\,x(0) + \cdots$$
$$= \underbrace{\left[I + At + \frac{A^2}{2}t^2 + \frac{A^3}{3!}t^3 + \cdots\right]}_{\text{define this as } \mathrm{e}^{At}} x(0).$$

$$x(t) = \mathrm{e}^{At}\,x(0) \qquad \text{and so} \qquad \Phi(t) = \mathrm{e}^{At}.$$

Properties:

$$\mathrm{e}^{A\times 0} = I, \quad \mathrm{e}^{A(s+t)} = \mathrm{e}^{As}\mathrm{e}^{At}, \quad \mathrm{e}^{-At}\mathrm{e}^{At} = I, \quad \frac{d\,\mathrm{e}^{At}}{dt} = A\mathrm{e}^{At}.$$

Solution via matrix exponential

We can begin by "guessing" a solution of the form,

$x(t) = \mathrm{e}^{At}\,v(t)$, where $v(t)$ is a time-varying vector.

Differentiate,

$$\frac{dx(t)}{dt} = A\mathrm{e}^{At}\,v(t) + \mathrm{e}^{At}\,\frac{dv(t)}{dt}$$
$$= A\,x(t) + B\,u(t) = A\mathrm{e}^{At}\,v(t) + B\,u(t).$$

So $\mathrm{e}^{At}\,\dfrac{dv(t)}{dt} = B\,u(t) \implies \dfrac{dv(t)}{dt} = \mathrm{e}^{-At}\,Bu(t).$

Solve this by integrating to get:

$$v(t) - v(0) = \int_0^t \mathrm{e}^{-A\tau}\,Bu(\tau)\,d\tau.$$

Solution via matrix exponential

$$v(t) - v(0) \;=\; \int_0^t \mathrm{e}^{-A\tau}\, Bu(\tau)\, d\tau.$$

and our assumed solution is:

$$x(t) \;=\; \mathrm{e}^{At}\, v(t) \quad \text{so} \quad v(t) \;=\; \mathrm{e}^{-At}\, x(t) \quad \text{and} \quad v(0) \;=\; x(0).$$

Substituting these gives,

$$\mathrm{e}^{-At}\, x(t) - x(0) \;=\; \int_0^t \mathrm{e}^{-A\tau}\, Bu(\tau)\, d\tau.$$

or,

$$\begin{aligned} x(t) \quad &= \quad \mathrm{e}^{At}\, x(0) \;+\; \mathrm{e}^{At} \int_0^t \mathrm{e}^{-A\tau}\, Bu(\tau)\, d\tau, \\ &= \quad \mathrm{e}^{At}\, x(0) \;+\; \int_0^t \mathrm{e}^{A(t-\tau)}\, Bu(\tau)\, d\tau, \end{aligned}$$

Again we have $\quad \Phi(t) = \mathrm{e}^{At}$.

Example

$$P(s) \quad = \quad \frac{(s-1)}{(s+1)(s+2)}.$$

This system has a state-space representation:

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad D = 0.$$

$$\begin{aligned} \mathrm{e}^{At} &= \mathcal{L}^{-1} \left\{ \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \right)^{-1} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s(s+3)+2} \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix} \right\} \\ &= \begin{bmatrix} 2\mathrm{e}^{-2t} - \mathrm{e}^{-t} & 2\mathrm{e}^{-2t} - 2\mathrm{e}^{-t} \\ -\mathrm{e}^{-2t} + \mathrm{e}^{-t} & -\mathrm{e}^{-2t} + 2\mathrm{e}^{-t} \end{bmatrix} \end{aligned}$$

Time-domain solutions

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau$$
$$= \mathsf{e}^{At}\,x(0) + \int_0^t \mathsf{e}^{A(t-\tau)}\,Bu(\tau)\,d\tau$$

Output solution

$$y(t) = Cx(t) + Du(t)$$
$$= C\mathsf{e}^{At}x(0) + C\int_0^t \mathsf{e}^{A(t-\tau)}\,Bu(\tau)\,d\tau + Du(t)$$

Impulse response

For the impulse response, $x(0) = 0$ and $u(t) = \delta(t)$,

$$g(t) = \begin{cases} 0 & t < 0 \\ C\mathsf{e}^{At}B + D\delta(t) & t \geq 0 \end{cases}$$

Transfer function representation

Take $x(t)\mid_{t=0} = x(0) = 0$.

$$sX(s) = AX(s) + BU(s) \quad \Longrightarrow \quad X(s) = (sI-A)^{-1}BU(s)$$
$$Y(s) = CX(s) + DU(s) \quad = \quad \underbrace{\left(C(sI-A)^{-1}B + D\right)}_{G(s)}\,U(s)$$
$$= \frac{1}{\det(sI-A)}(C\ \mathrm{adj}(sI-A)\,B)\ +\ D$$

Notation

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Example (continued)

$$P(s) \quad = \quad \frac{(s-1)}{(s+1)(s+2)} \quad = \quad \left[\begin{array}{cc|c} -3 & -2 & 1 \\ 1 & 0 & 0 \\ \hline 1 & -1 & 0 \end{array}\right].$$

From before we have,

$$(sI-A)^{-1} = \frac{1}{s(s+3)+2}\begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix},$$

so,

$$\begin{aligned} C(sI-A)^{-1}B + D &= \frac{1}{s(s+3)+2}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \\ &= \frac{1}{s(s+3)+2}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} s \\ 1 \end{bmatrix} \\ &= \frac{s-1}{s^2+3s+2}. \end{aligned}$$

Transfer function representations

Controllable canonical form

One possible state-space representation of a transfer function.

For example,

$$G(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

Controllable canonical form:

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}, \qquad D = 0$$

System poles

For a system, $G(s)$, with a (simple) pole at $s = p_i$,

$$G(s) = \frac{a(s)}{b(s)} = \frac{(s - z_1)\dots(s - z_m)}{(s - p_1)\dots(s - p_i)\dots(s - p_n)}$$

$$= \frac{E_1}{(s - p_1)} + \cdots + \frac{E_i}{(s - p_i)} + \cdots + \frac{E_n}{(s - p_n)}.$$

Impulse response;

$$g(t) = E_1 e^{p_1 t} + \cdots + E_i e^{p_i t} + \cdots + E_n e^{p_n t}.$$

The zero-input solutions of the corresponding differential equation will have terms of the form,

$$y(t) \quad = \quad k_i e^{p_i t} \ + \ \dots$$

System poles

Zero-input case: $\frac{dx(t)}{dt} = A\,x(t)$

$$x(t) \quad = \quad e^{p_i t} x(0) \qquad \longleftarrow \qquad \text{form of candidate solution.}$$

Differentiating,

$$\frac{dx(t)}{dt} \quad = \quad p_i e^{p_i t} x_0 \quad = \quad p_i\,x(t) \quad = \quad Ax(t).$$

Choosing $t = 0$ gives,

$$A\,x(0) \quad = \quad p_i\,x(0) \qquad \longleftarrow \qquad \text{eigenvalue equation}$$

The eigenvalues of A are the poles of $P(s)$.

The poles (eigenvalues) are also called "natural frequencies" or "modes" of $G(s)$.

System zeros

If $G(s)$ has a zero at $s = s_0$,

There exists $u(s_0) \neq 0$ and $x(s_0) \neq 0$, such that $G(s_0)u(s_0) = 0$.

State-space Laplace domain (with $s = s_0$),

$$\begin{aligned} s_0 x(s_0) &= A\,x(s_0) + B\,u(s_0) \\ 0 &= C\,x(s_0) + D\,u(s_0) \end{aligned}$$

In matrix form:

$$\begin{bmatrix} (s_0 I - A) & -B \\ C & D \end{bmatrix} \begin{bmatrix} x(s_0) \\ u(s_0) \end{bmatrix} = 0, \qquad \text{with the constraint: } x(s_0) \neq 0, u(s_0) \neq 0.$$

Or equivalently,

$$\det \begin{bmatrix} (s_0 I - A) & -B \\ C & D \end{bmatrix} = 0 \qquad \text{with the constraint: } x(s_0) \neq 0, u(s_0) \neq 0.$$

System zeros

At a zero $s = z_i$, the rank of $G(s)\,|_{s=z_i}$ drops from it's "normal" value.

Examples

$$G_1(s) = \begin{bmatrix} \frac{(s+2)}{(s+1)} & 0 \\ 0 & \frac{(s+1)}{(s+2)} \end{bmatrix}$$

- $G_1(s)$ has poles at -1 and -2
- $G_1(s)$ has zeros at -1 and -2
- Poles and zeros do not cancel (different directions)

$$G_2(s) = \begin{bmatrix} \frac{(s+2)}{(s+1)} \\ \frac{(s+1)}{(s+2)} \end{bmatrix} \qquad G_3(s) = \begin{bmatrix} \frac{(s+2)}{(s+1)} & \frac{(s+1)}{(s+2)} \end{bmatrix}$$

- $G_2(s)$ and $G_3(s)$ have poles at -1 and -2
- $G_2(s)$ has no MIMO zeros.
- $G_3(s)$ has no MIMO zeros.

Similarity transformations

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \qquad A \in \mathcal{R}^{n\times n}$$
$$y(t) = Cx(t) + Du(t) \qquad D \in \mathcal{R}^{n_y\times n_u}$$

Define a new state, $z = Tx$, with T invertible.

$$T^{-1}\frac{dz(t)}{dt} = AT^{-1}z(t) + Bu(t)$$
$$y(t) = CT^{-1}z(t) + Du(t)$$

Giving the equivalent system with state z(t)

$$\frac{dz(t)}{dt} = TAT^{-1}z(t) + TBu(t)$$
$$y(t) = CT^{-1}z(t) + Du(t)$$

Other domains

Continuous time, time invariant:

$$\begin{aligned}\frac{d\,x(t)}{dt} &= A\,x(t) + B\,u(t),\\ y(t) &= C\,x(t) + D\,u(t)\end{aligned}$$

Discrete time, time invariant:

$$\begin{aligned}x(k+1) &= A\,x(k) + B\,u(k),\\ y(k) &= C\,x(k) + D\,u(k)\end{aligned}$$

Nonlinear, time invariant:

$$\begin{aligned}\frac{d\,x(t)}{dt} &= f(x(t), u(t)),\\ y(t) &= g(x(t), u(t))\end{aligned}$$

Nonlinear, time varying:

$$\begin{aligned}\frac{d\,x(t)}{dt} &= f(t, x(t), u(t)),\\ y(t) &= g(t, x(t), u(t))\end{aligned}$$

State controllability

Formal definition:

A system is *controllable* (or *state controllable*) if and only if for any given initial state, $x(t_0) = x_0$, and any given final state, x_1, specified at an arbitrary future time, $t_1 > t_0$, there exists a control input $u(t)$ that will drive the system from $x(t_0) = x_0$ to $x(t_1) = x_1$.

Controllability matrix:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathcal{R}^{n \times n_u n}.$$

A system is called *"controllable"* if and only if $\mathcal{C}$ has rank n (equivalently for $n_u = 1$, $\mathcal{C}^{-1}$ exists).

State controllability

Controllability Grammian

$$W_c(t) := \int_0^t \mathrm{e}^{A\tau} BB^T \mathrm{e}^{A^T\tau}\, d\tau \quad \in \mathcal{R}^{n \times n}.$$

To get from state $x(t_0)$ to $x(t_1)$, (with $t_1 > t_0$), use,

$$u(t) = -B^T \mathrm{e}^{A^T(t_1 - t)} W_c^{-1}(t_1) \left(\mathrm{e}^{At_1} x_0 - x_1 \right).$$

(A,B) is controllable if and only if $W_c(t)$ has rank n for all $t > 0$.

Lyapunov equation

$$P := W_c(\infty) = \int_0^\infty \mathrm{e}^{A\tau} BB^T \mathrm{e}^{A^T\tau}\, d\tau. \qquad \text{(for stable systems)}$$

We can show that P solves the Lyapunov equation:

$$AP + PA^T = -BB^T.$$

State controllability

Controllability as a design tool

Consider a two-input, two-output system,

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

This system is unstable (where are the poles?)

Suppose that implementing $u_1(t)$ costs CHF 50 and implementing $u_2(t)$ costs CHF 100. What is the minimum cost to stabilize the system: CHF 50, CHF 100 or CHF 150?

State observability

Formal definition:

The dynamical system pair (A,C) is state observable if and only if, for any time $t_1 > 0$, the initial state, $x(0)$ can be determined from the time history of the input, $u(t)$, and output, $y(t)$, over the interval $[0, t_1]$.

Observability matrix

$$(A, C) \text{ is observable} \iff \operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

State observability

Observability Grammian:

Observability (for a stable system) is equivalent to

$$Q := \int_0^\infty \mathrm{e}^{A^T \tau} C^T C \mathrm{e}^{A\tau}\, d\tau,$$

having full rank (positive definite).

Lyapunov equation

Q solves,

$$A^T Q + QA = -C^T C.$$

Kalman decomposition

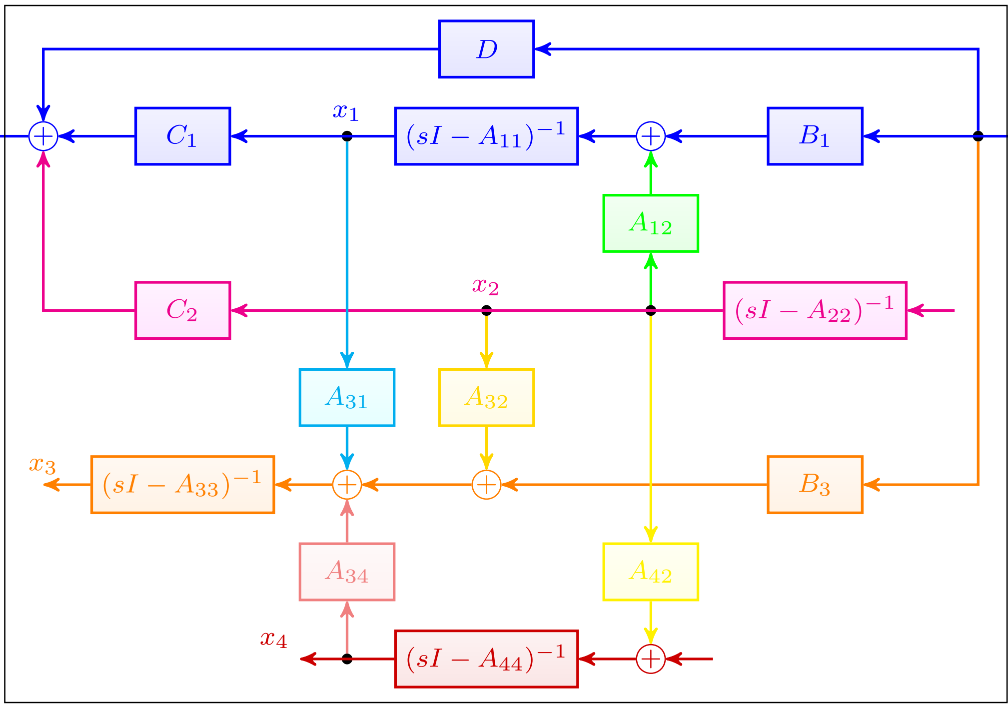
$$G(s) = \left[\begin{array}{cccc|c} A_{11} & A_{12} & 0 & 0 & B_1 \\ 0 & A_{22} & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & B_3 \\ 0 & A_{42} & 0 & A_{44} & 0 \\ \hline C_1 & C_2 & 0 & 0 & D \end{array}\right]$$

States	**Decomposition**
x_1	controllable and observable
x_2	uncontrollable and observable
x_3	controllable and unobservable
x_4	uncontrollable and unobservable

A system is stabilizable if all of its unstable modes are state controllable.

A system is detectable if all of its unstable modes are state observable.

Kalman decomposition



Notes and references

Skogestad & Postlethwaite (2nd Ed.)

State space systems: section 4.1

Poles and zeros: sections 4.4 & 4.5

Controllability and observability: section 4.2

Stability: sections 4.3 & 4.7