Control Systems 2

Lecture 5: RHP poles and zero limitations
& how to design and ride a bike

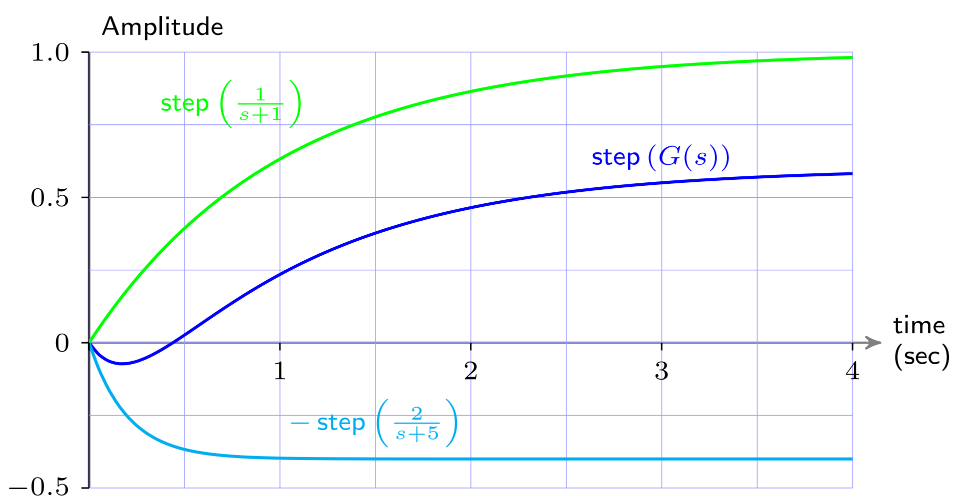
Roy Smith

Non-minimum phase behaviour (stable systems)

Right-half plane zeros

Can arise from fast and slow responses of opposite sign:

$$G(s) = \frac{1}{s+1} - \frac{2}{s+5} = \frac{3-s}{(s+1)(s+5)}.$$

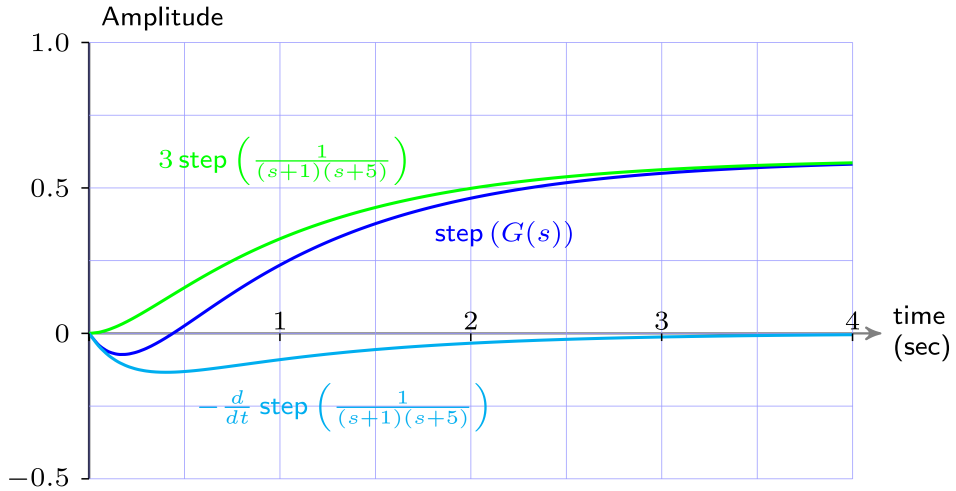


Non-minimum phase behaviour

Can also be interpreted as a negative derivative response:

$$G(s) = \frac{3}{(s+1)(s+5)} - \frac{s}{(s+1)(s+5)}$$

$$g(t) = 3\left(\frac{1}{4}\mathrm{e}^{-t} + \frac{-1}{4}\mathrm{e}^{-5t}\right) - \frac{d}{dt}\left(\frac{1}{4}\mathrm{e}^{-t} + \frac{-1}{4}\mathrm{e}^{-5t}\right)$$



Non-minimum phase systems

Some common examples

- Longitudinal aircraft dynamics
- Human digestion (energy from food)
- Investment effects on profitability
- Bicycle steering dynamics

Non-minimum phase systems in feedback

Non-minimum phase response in closed-loop

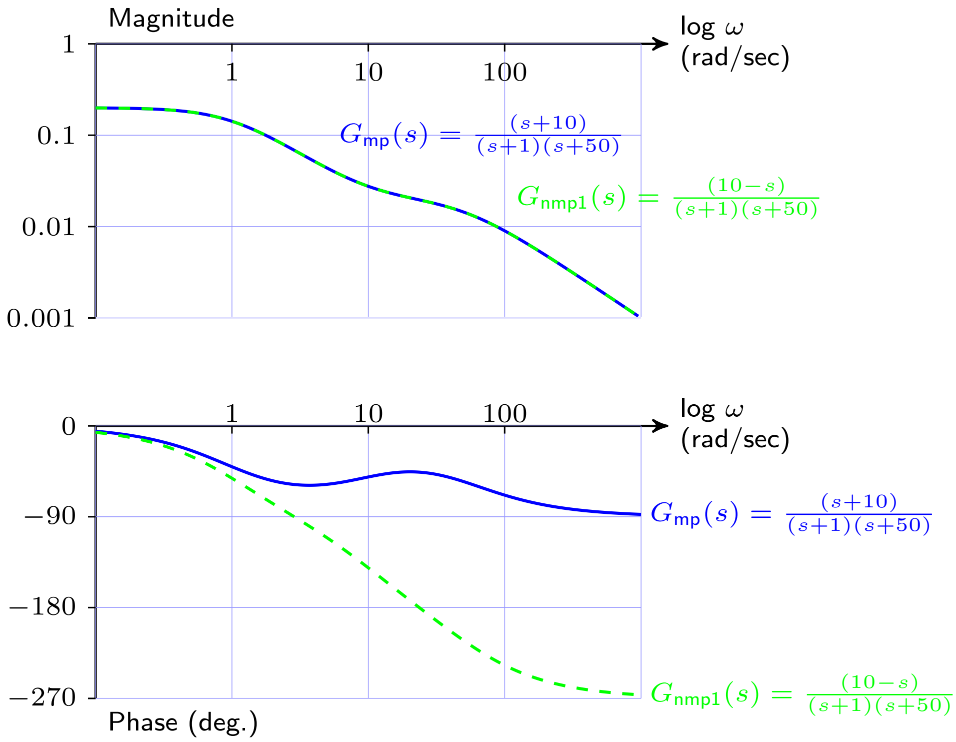
$$G(s) = \frac{N_G(s)}{D_G(s)}, \qquad K(s) = \frac{N_K(s)}{D_K(s)}, \qquad L(s) = \frac{N_G(s)N_K(s)}{D_G(s)D_K(s)}$$

$$T(s) = \frac{L(s)}{1 + L(s)}$$

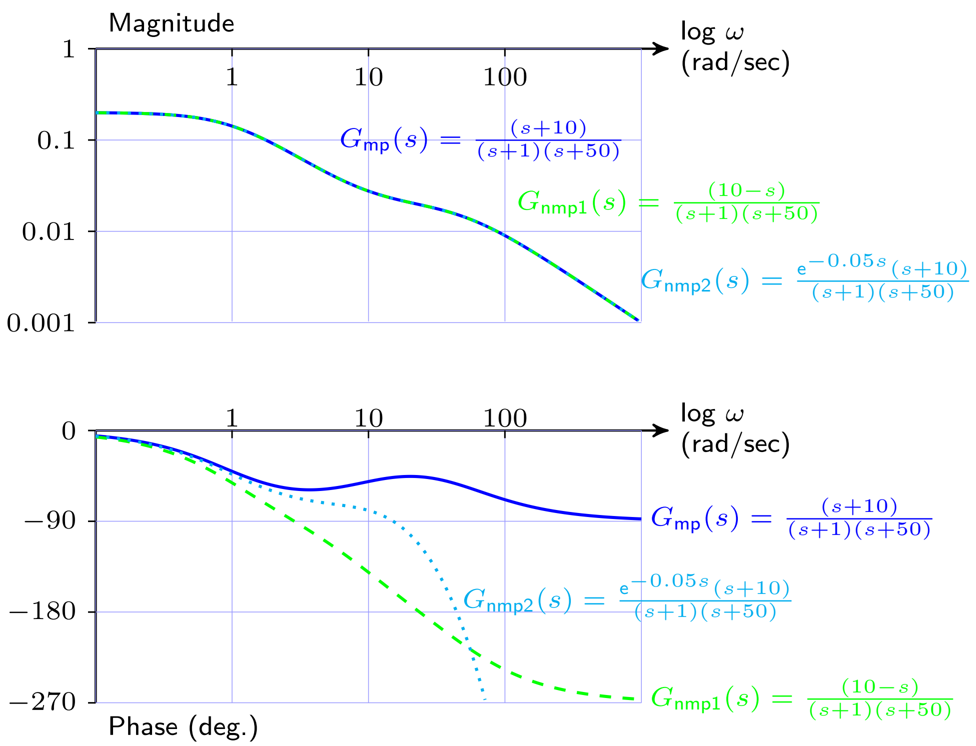
$$= \frac{\frac{N_G(s)}{D_G(s)}\frac{N_K(s)}{D_K(s)}}{1 + \frac{N_G(s)}{D_G(s)}\frac{N_K(s)}{D_K(s)}}$$

$$= \frac{N_G(s)N_K(s)}{D_G(s)D_K(s) + N_G(s)N_K(s)}$$

Non-minimum phase systems: r.h.p. zeros



Non-minimum phase systems: delays



Non-minimum phase systems in feedback

Delays in feedback

$$G(s) = \mathrm{e}^{-\theta s}\frac{N_G(s)}{D_G(s)}, \quad K(s) = \frac{N_K(s)}{D_K(s)} \quad L(s) = \mathrm{e}^{-\theta s}\frac{N_G(s)N_K(s)}{D_G(s)D_K(s)}$$

$$T(s) = \frac{L(s)}{1+L(s)}$$

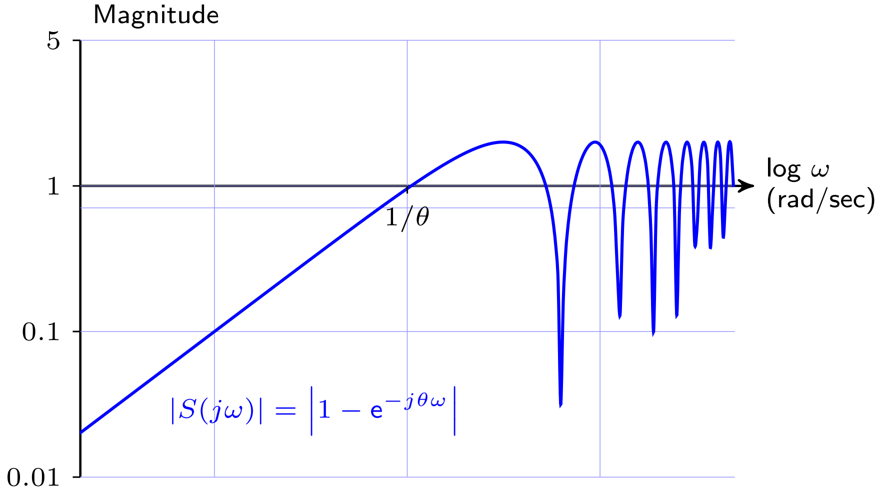
$$= \frac{\mathrm{e}^{-\theta s}\frac{N_G(s)}{D_G(s)}\frac{N_K(s)}{D_K(s)}}{1+\mathrm{e}^{-\theta s}\frac{N_G(s)}{D_G(s)}\frac{N_K(s)}{D_K(s)}}$$

$$= \mathrm{e}^{-\theta s}\left(\frac{N_G(s)N_K(s)}{D_G(s)D_K(s)+\mathrm{e}^{-\theta s}N_G(s)N_K(s)}\right)$$

Performance limitations from delays

If $G(s)$ contains a delay, $\mathrm{e}^{-\theta s}$, then $T(s)$ also contains $\mathrm{e}^{-\theta s}$.

Under these circumstances the ideal $T(s) \approx \mathrm{e}^{-\theta s}$,



Which implies that we must have $\omega_c < 1/\theta$.

Controllability (summary)

Actuation constraints: from disturbances

$|G(j\omega)| > |G_\mathrm{d}(j\omega)|$ for frequencies where $|G_\mathrm{d}(j\omega)| > 1$.

Actuation constraints: from reference

$|G(j\omega)| > R$ up to frequency: ω_r.

Disturbance rejection

$\omega_c > \omega_d$

or more specifically $|S(j\omega)| \leq |1/G_\mathrm{d}(j\omega)|$ for all ω.

Reference tracking

$|S(j\omega)| \leq 1/R$ up to frequency: ω_r.

Controllability (summary)

Right-half plane zeros

For a single, real, RHP-zero: $\omega_B < z/2$.

Time delays

Approximately require: $\omega_c < 1/\theta$.

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Most practical controllers (PID/lead-lag): $\omega_c < \omega_{180}$
$G(j\omega_{180}) = -180$ deg.

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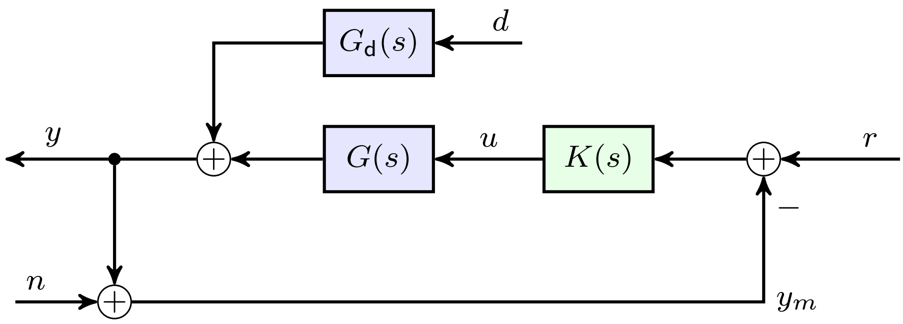
Unstable real pole

Require $\omega_c > 2p$.
Also require $|G(j\omega)| > |G_\mathsf{d}(j\omega)|$ up to $\omega = p$.

Example: controllability analysis



$$G(s) = k\frac{\mathrm{e}^{-\theta s}}{1+\tau s} \qquad G_\mathsf{d}(s) = k_d\frac{\mathrm{e}^{-\theta_d s}}{1+\tau_d s}, \quad |k_d| > 1.$$

What are the requirements on k, k_d, τ, τ_d, θ and θ_d in order to obtain good performance. And how good is it?

Example: controllability analysis

Objective:

$|e| \leq 1$ for all $|u| < 1,\ |d| < 1.$

Disturbance rejection (satisfying actuation bound)

$|G(j\omega)| > |G_\mathrm{d}(j\omega)|$ for all $\omega < \omega_d$.

$\implies k > k_d$ and $k/\tau > k_d/\tau_d$.

Disturbance rejection

$\omega_c > \omega_d \approx k_d/\tau_d$.

Delay constraints

$\omega_c < 1/\theta$ (assuming θ is the total delay in the loop).

Example: controllability analysis

Delay and disturbance rejection requirements.

$\theta < \tau_d/k_d$.

Plant requirements:

$k > k_d$ and $k/\tau > k_d/\tau_d$

$\theta < \tau_d/k_d$.

Required/achievable bandwidth

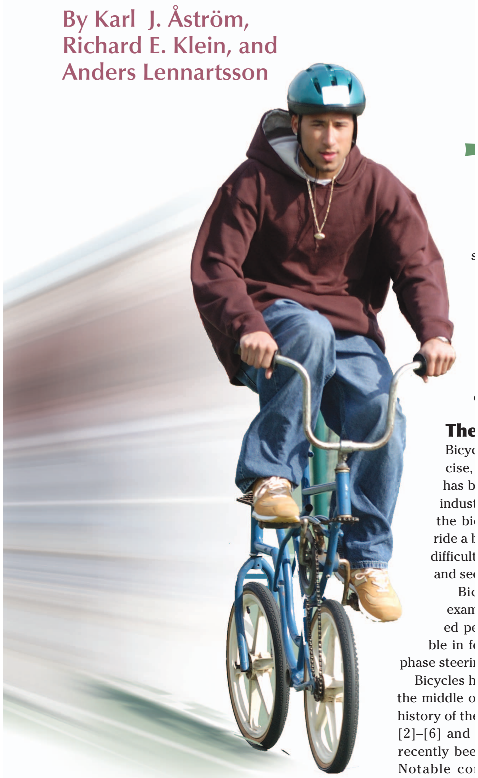
$k_d/\tau_d < \omega_c < 1/\theta$.

Bicycle dynamics

Bicycle Dynamics and Control

Adapted bicycles for
education and research

IEEE Control Systems Magazine
vol. 25, no. 4, pp. 26–47,
2005.



Bike parameter definitions

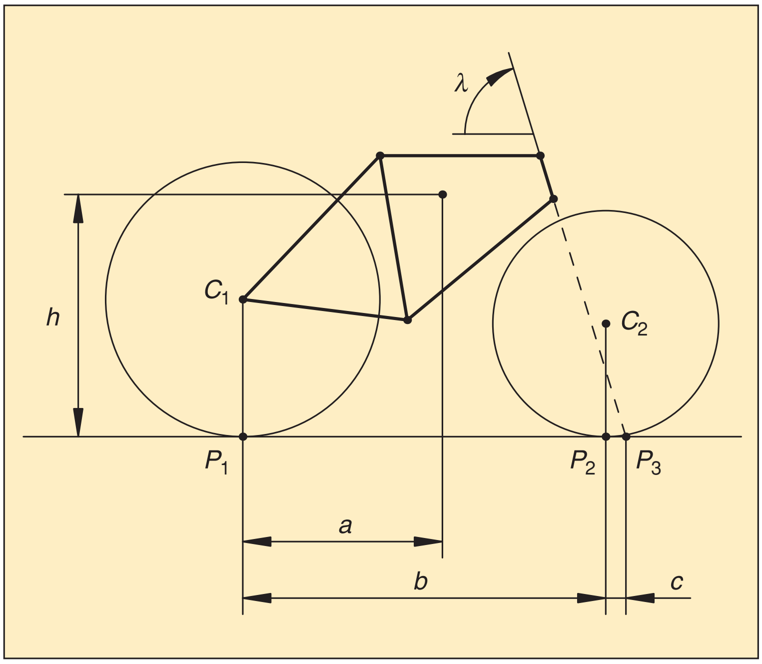


Figure 1. *Parameters defining the bicycle geometry. The points P_1 and P_2 are the contact points of the wheels with the ground, the point P_3 is the intersection of the steer axis with the horizontal plane, a is the distance from a vertical line through the center of mass to P_1, b is the wheel base, c is the trail, h is the height of the center of mass, and λ is the head angle.*

Reference frame definitions

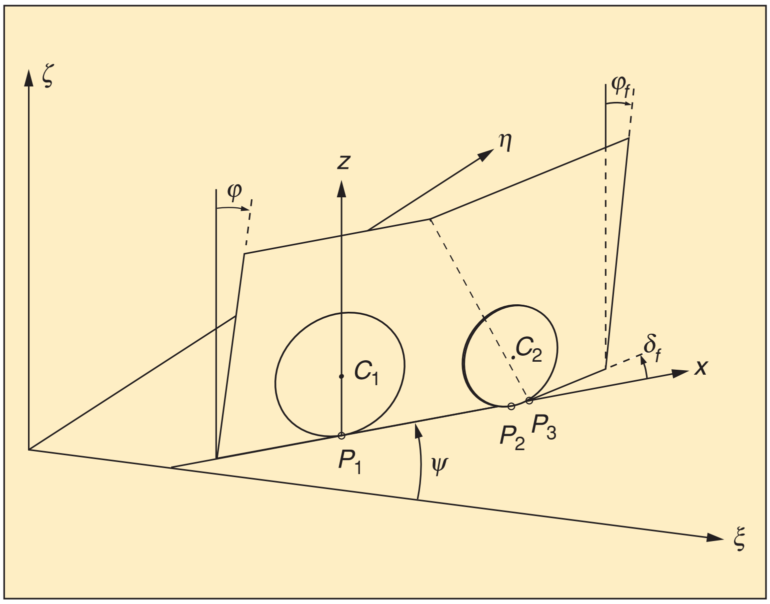


Figure 2. *Coordinate systems. The orthogonal system $\xi\eta\zeta$ is fixed to inertial space, and the ζ-axis is vertical. The orthogonal system xyz has its origin at the contact point of the rear wheel with the $\xi\eta$ plane. The x axis passes through the points P_1 and P_3, while the z axis is vertical and passes through P_1.*

Naïve analysis

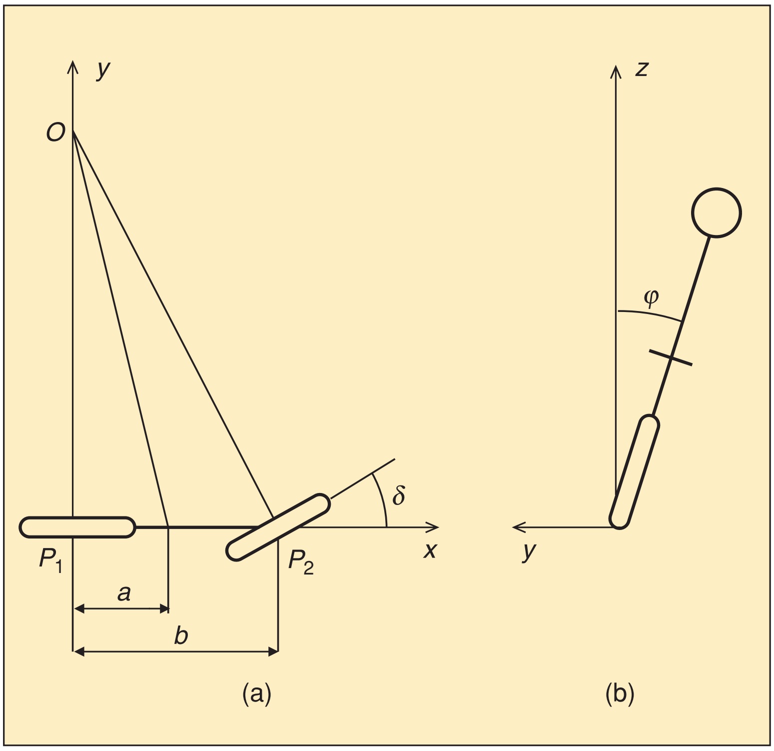


Figure 3. *Schematic (a) top and (b) rear views of a naive ($\lambda = 0$) bicycle. The steer angle is δ, and the roll angle is φ.*

Typographical error: $\lambda = 90$.

Naïve analysis: simple second order models

Steering angle, δ, to tilt angle, ϕ, transfer function

$L_x = J\frac{d\phi}{dt} - D\omega = J\frac{d\phi}{dt} - \frac{VD}{b}\delta$	Angular momentum about x
$J\frac{d^2\phi}{dt^2} - mgh\phi = \frac{DV}{b}\frac{d\delta}{dt} + \frac{mV^2h}{b}\delta$	Torque balance
$J \approx mh^2$ and $D \approx mah$	Inertia approximations
$\frac{d^2\phi}{dt^2} - \frac{g}{h}\phi = \frac{aV}{bh}\frac{d\delta}{dt} + \frac{V^2}{bh}\delta$	Simplified model

Naïve analysis: simple second order models

Steering angle, δ, to tilt angle, ϕ, transfer function

Transfer function:

$$G_{\phi\delta}(s) = \frac{\phi(s)}{\delta(s)} = \frac{V(Ds + mVh)}{b(Js^2 - mgh)} \approx \frac{aV}{bh}\,\frac{(s + V/a)}{(s^2 - g/h)}$$

Naïve analysis: simple second order models

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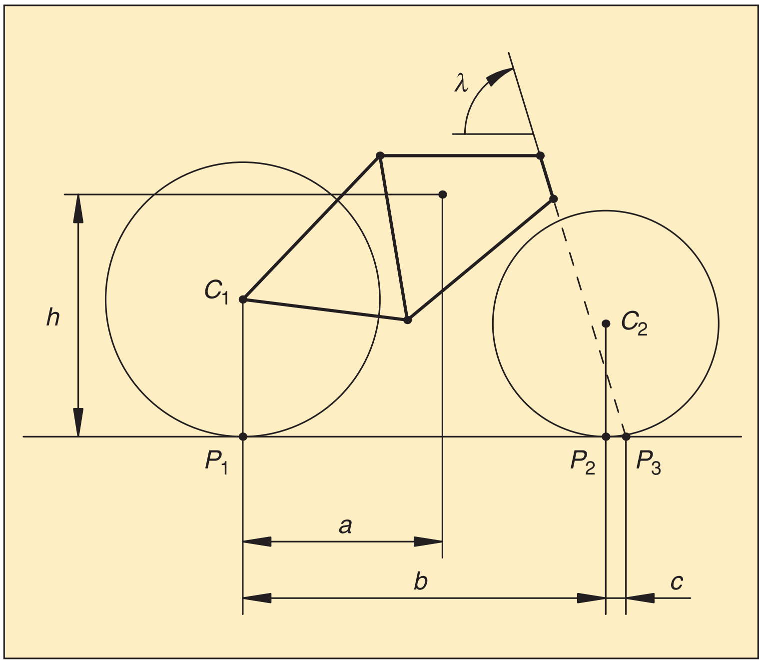
poles: $p_{1,2} = \pm\sqrt{\dfrac{mgh}{J}} \approx \pm\sqrt{\dfrac{g}{h}}$

zero: $z_1 = -\dfrac{mVh}{D} \approx -\dfrac{V}{a}$

Bike parameter definitions



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Front fork model

Handlebar torque, T, to tilt angle, ϕ, transfer function

Model the actuation as a torque to the handlebars, T.

$$J\frac{d^2\phi}{dt^2} + \frac{DVg}{V^2\sin\lambda - bg\cos\lambda}\frac{d\phi}{dt} + \frac{mg^2(bh\cos\lambda - ac\sin\lambda)}{V^2\sin\lambda - bg\cos\lambda}\phi$$

$$= \frac{DVb}{acm(V^2\sin\lambda - bg\cos\lambda)}\frac{dT}{dt} + \frac{b(V^2h - acg)}{ac(V^2\sin\lambda - bg\cos\lambda)}T$$

The system is stable if $V > V_c = \sqrt{bg\cot\lambda}$ and $bh > ac\tan\lambda$

Gyroscopic effects could be included (giving additional damping).

Front fork model

Torque to steering angle transfer function

With a stabilizable bicycle going at sufficiently high speed, V,

$$\frac{\delta}{T} = G_{\delta T}(s) = \frac{k_1(V)}{1 + k_2(V)G_{\phi\delta}(s)},$$

where, as before, $G_{\phi\delta}(s) = \frac{V(Ds + mVh)}{b(Js^2 - mgh)} \approx \frac{aV}{bh}\frac{(s + V/a)}{(s^2 - g/h)}$

So, $$G_{\delta T}(s) = \frac{k_1(V)\left(s^2 - \frac{mgh}{J}\right)}{s^2 + \frac{k_2(V)DV}{bJ}s + \frac{k_2(V)V^2mh}{bJ} - \frac{mgh}{J}}$$

Front fork model

Torque to path deviation transfer function

If η is the deviation in path,

$$G_{\eta T}(s) = \frac{k_1(V)\,V^2}{b} \frac{\left(s^2 - \frac{mgh}{J}\right)}{s^2\left(s^2 + \frac{k_2(V)DV}{bJ}\,s + \frac{mgh}{J}\left(\frac{V^2}{V_c^2} - 1\right)\right)}$$

Non-minimum phase behaviour

Counter-steering

"I have asked dozens of bicycle riders how they turn to the left. I have never found a single person who stated all the facts correctly when first asked. They almost invariably said that to turn to the left, they turned the handlebar to the left and as a result made a turn to the left. But on further questioning them, some would agree that they first turned the handlebar a little to the right, and then as the machine inclined to the left they turned the handlebar to the left, and as a result made the circle inclining inwardly."
Wilbur Wright.

Non-minimum phase behaviour

Counter-steering

Non-minimum phase behaviour

Aircraft control

"Men know how to construct airplanes. Men also know how to build engines. Inability to balance and steer still confronts students of the flying problem. When this one feature has been worked out, the age of flying will have arrived, for all other difficulties are of minor importance."
Wilbur Wright, 1901.

Rear-wheel steered bicycles

Rear-wheel steered bicycles

Stabilization: simple model

The sign of V is reversed in all of the equations.

$$G_{\phi\delta}(s) = \frac{-VDs + mV^2h}{b(Js^2 - mgh)} = \frac{VD}{bJ}\frac{\left(-s + \frac{mVh}{D}\right)}{\left(s^2 - \frac{mgh}{J}\right)}$$

$$\approx \frac{aV}{bh}\frac{(-s + V/a)}{(s^2 - g/h)}$$

Rear-wheel steered bicycles

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$$\approx \frac{aV}{bh}\,\frac{(-s + V/a)}{(s^2 - g/h)}$$

This now has a RHP pole and a RHP zero.

The zero/pole ratio is: $\frac{z}{p} = \frac{mVh}{D}\sqrt{\frac{J}{mgh}} \approx \frac{V}{a}\sqrt{\frac{h}{g}}$

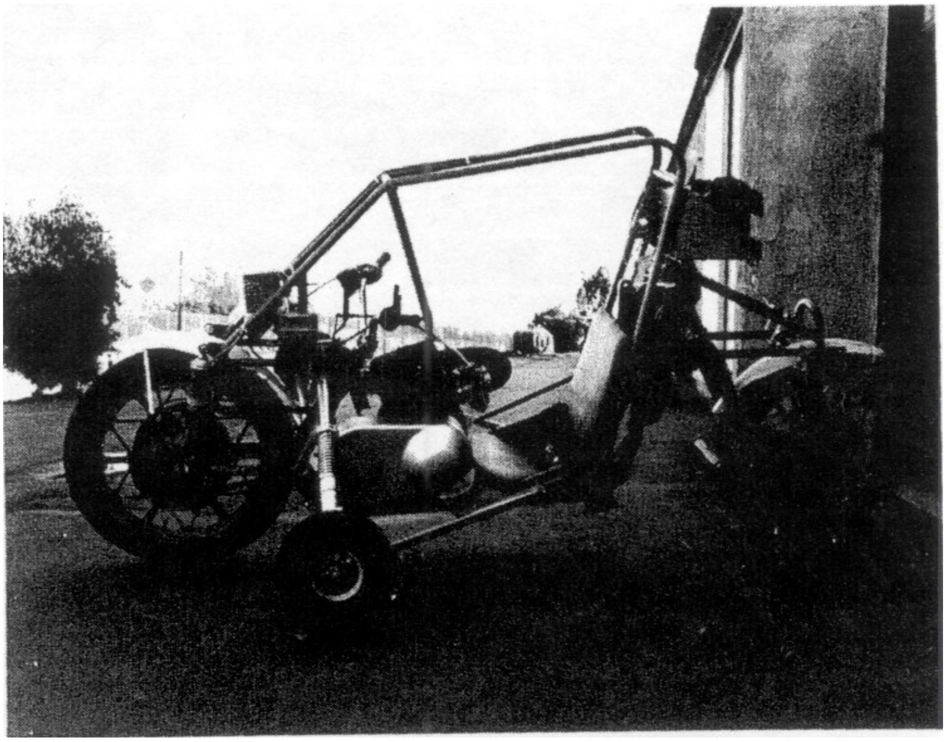
Rear-wheel steered motorbikes

NHSA Rear-steered Motorcycle

- 1970's research program sponsored by the US National Highway Safety Administration.
- Rear steering benefits: Low center of mass.
  Long wheel base.
  Braking/steering on different wheels
- Design, analysis and building by South Coast Technologies, Santa Barbara, CA.
- Theoretical study: $\mathrm{real}(p)$ in range 4 – 12 rad/sec. for V of 3 – 50 m/sec.
- Impossible for a human to stabilize.

Rear-wheel steered motorbikes

NHSA Rear-steered Motorcycle



Rear-wheel steered motorbikes

NHSA Rear-steered Motorcycle

"The outriggers were essential; in fact, the only way to keep the machine upright for any measurable period of time was to start out down on one outrigger, apply a steer input to generate enough yaw velocity to pick up the outrigger, and then attempt to catch it as the machine approached vertical. Analysis of film data indicated that the longest stretch on two wheels was about 2.5 seconds."
Robert Schwartz, South Coast Technology, 1977.

Rear-wheel steered motorbikes

Meeks' bike: "Quantum Leap"

www.autoevolution.com

www.robbreport.com

Rear-wheel steered motorbikes

Meeks' bike: "Quantum Leap"

www.autoevolution.com

www.robbreport.com

Meeks' reason for not riding it

"The bike's so expensive, it's a concept that's going to be shown and to ride it and to take a chance of chipping or scratching it, it's not worth it. All we wanted to do was make sure it worked, which we did."

Rear-wheel steered bicycles

UCSB bike

Rear-wheel steered bicycles

An unridable bike

Rear-wheel steered bicycles

This one had another problem!

Notes and references

Skogestad & Postlethwaite (2nd Ed.)

Control limitations: sections 5.6 – 5.11

Practical controllability examples: sections 5.13 – 5.15.

More on bicycles

TU Delft: `http://bicycle.tudelft.nl/schwab/Bicycle/index.htm`

Veritasium “Most people don’t know how bicycles actually work,” `https://www.youtube.com/watch?v=9cNmUNHSBac`

Article: Karl J. Åström, Richard E. Klein & Anders Lennartsson, “Bicycle dynamics and control,” *IEEE Control Systems Magazine*, vol. 25, no. 4, pp. 26–47, 2005.