

Control Systems 2

Lecture 4: Sensitivity function limits

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Input-output controllability

Control design questions:

1. How well can the plant be controlled?
2. What control structure should be used?
3. How might the process be changed in order to improve the control?

Ideally we want some idea of the answers to these questions without an exhaustive trial-and-error design approach.

Input-output controllability

Can we achieve the control design specifications?

- ▶ ... with limited actuation authority;
- ▶ ... in the presence of bounded disturbances;
- ▶ ... in the presence of sensor noise;
- ▶ ... in the presence of plant variations;
- ▶ ... without exact knowledge of the plant.

These questions depend on the properties of the plant and the control resources (gain, bandwidth, resolution, computation) available.

Scaling

Scale the reference by $R(\omega)$:

Define the error as $e = y - R(\omega)r$ with $|r(\omega)| \leq 1$.

Scale $G(s)$ and $G_d(s)$ such that:

Control objective

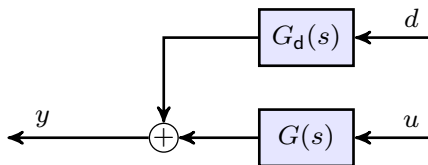
For any disturbance, $|d(\omega)| \leq 1$, and any reference, $|r(\omega)| \leq 1$, ...

make $|e(\omega)| \leq 1$ using an input $|u(\omega)| \leq 1$.

For example:

$$R(\omega) = \begin{cases} R & \text{if } \omega \leq \omega_r \\ 0 & \text{if } \omega > \omega_r \end{cases}$$

“Perfect” control



$$y = G(s)u + G_d(s)d$$

“Perfect” control would give: $y = Rr$ or $e = Rr - y = 0$.

$$e = Rr - Gu - G_d d$$

“Perfect” control (feedforward)

Error:

$$e = Rr - y = Rr - Gu - G_d d$$

Feedforward solution:

$$u = G^{-1}(s)Rr - G^{-1}(s)G_d(s)d$$

Limitations:

- ▶ $G(s)$ has r.h.p. zeros ($G^{-1}(s)$ is unstable).
- ▶ $G(s)$ has delays ($G^{-1}(s)$ is acausal).
- ▶ $G(s)$ is strictly proper ($G^{-1}(s)$ is unrealizable).
- ▶ d is not known, measured, or predicted.
- ▶ u will have a maximum control frequency.

“Perfect” control (feedforward)

Feedforward solution:

$$u = G^{-1}(s)Rr - G^{-1}(s)G_d(s)d$$

Control objective requirements:

$$|G^{-1}(j\omega)R(\omega)| \leq 1 \implies |G(j\omega)| \geq |R(\omega)| \text{ for all } \omega$$

$$|G^{-1}(j\omega)G_d(j\omega)| \leq 1 \implies |G(j\omega)| \geq |G_d(j\omega)| \text{ for all } \omega$$

“Perfect” control (feedback)

Feedback solution:

$$e = Rr - y = SRr - SG_d d \quad \text{Ideally } S(j\omega) = 0.$$

When $S \approx 0$, $T \approx 1$, and the control actuation, u , is approximately that generated by “perfect” feedforward control.

$$u = K(Rr - y) = KSRr - KSG_d d = G^{-1}TRr - G^{-1}TG_d d$$

“Perfect” control (feedback)

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Control advantages

Correct u (or close to correct) generated:

- ▶ ... without having to invert $G(s)$,
- ▶ ... without exact knowledge of $G(s)$,
- ▶ ... without knowing d .

Sensitivity/complementary sensitivity

$$\begin{aligned} S(s) + T(s) &= \frac{1}{1 + G(s)K(s)} + \frac{G(s)K(s)}{1 + G(s)K(s)} \\ &= \frac{1 + G(s)K(s)}{1 + G(s)K(s)} \\ &= 1 \quad \text{for all } s \in \mathcal{C}. \end{aligned}$$

Bode sensitivity integral

Given a loopshape, $L(s)$, with:

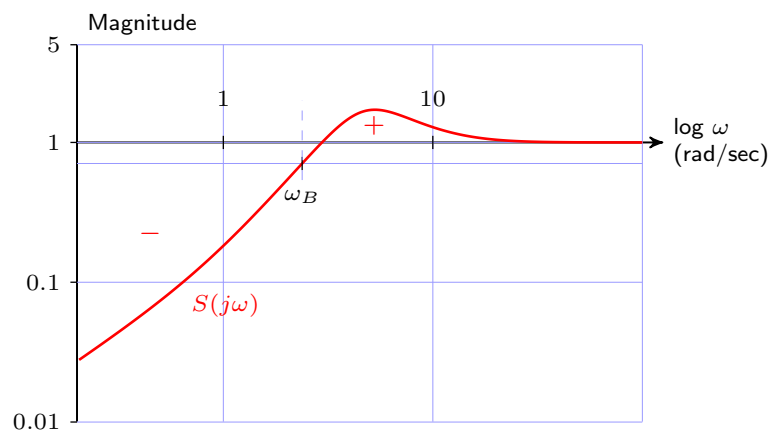
- relative degree of at least 2,
- N_p poles, p_i , in the right-half plane.

If the closed-loop is stable then,

$$\int_0^{\infty} \ln |S(j\omega)| d\omega = \begin{cases} \pi \sum_{i=1}^{N_p} \text{real}(p_i) & \text{if } L(s) \text{ is unstable} \\ 0 & \text{if } L(s) \text{ is also stable} \end{cases}$$

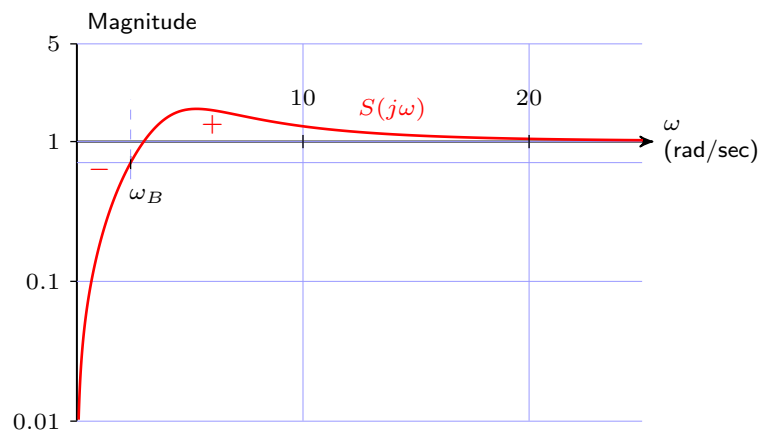
Bode sensitivity integral

$$\int_0^{\infty} \ln |S(j\omega)| d\omega = 0 \quad (\text{stable example})$$

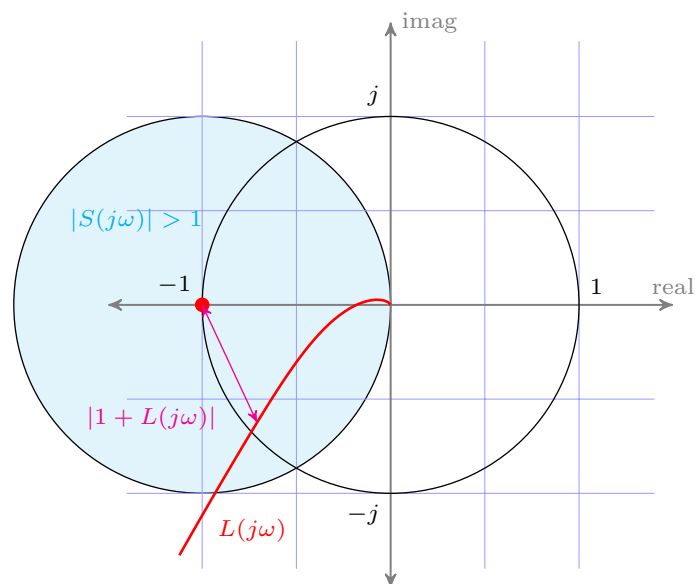


Bode sensitivity integral

$$\int_0^{\infty} \ln |S(j\omega)| d\omega = 0 \quad (\text{stable example})$$



Bode sensitivity integral



Bode sensitivity integral (with a right-half plane zero)

Given a loopshape, $L(s)$, with:

- $L(s)$ being proper,
- one right-half plane zero at $s = z_0$,
- N_p poles, p_i , in the right-half plane.

If the closed-loop is stable then,

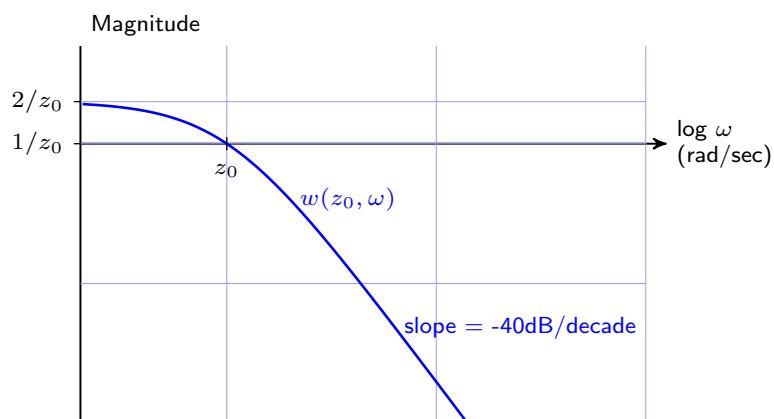
$$\int_0^\infty \ln |S(j\omega)| w(z_0, \omega) d\omega = \pi \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z_0}{p_i - z_0} \right|$$

where the “weighting” function is:

$$w(z_0, \omega) = \frac{2z_0}{z_0^2 + \omega^2} = \frac{2}{z_0} \frac{1}{(1 + (\omega/z_0)^2)}.$$

Bode sensitivity integral (with a right-half plane zero)

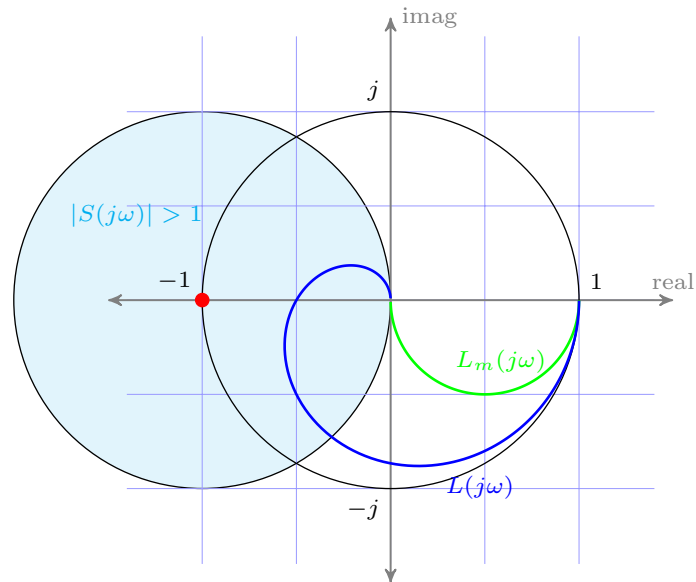
$$\int_0^\infty \ln |S(j\omega)| w(z_0, \omega) d\omega \approx \int_0^{z_0} \ln |S(j\omega)| d\omega = 0 \quad (\text{stable case})$$



- ▶ “bad” S performance must happen before $s = z_0$.
- ▶ Situation is worse with an unstable pole as well.

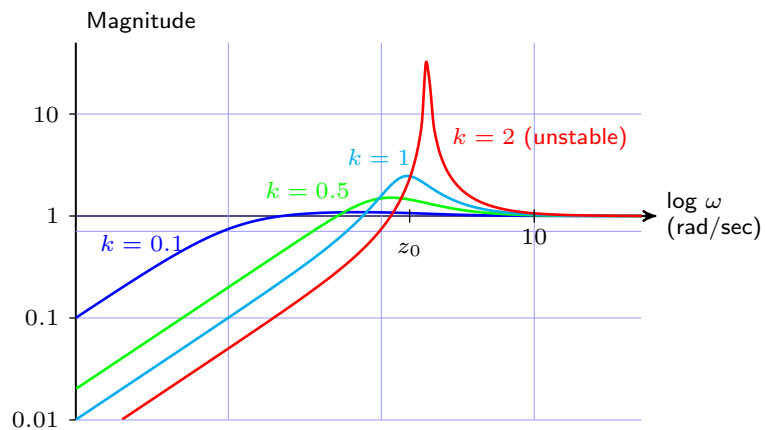
Bode sensitivity integral (with a right-half plane zero)

$$L_m(s) = \frac{1}{s+1} \quad L(s) = L_m(s) \frac{1-s}{s+1}$$



Bode sensitivity integral (with a right-half plane zero)

$$L(s) = \frac{k}{s} \left(\frac{2-s}{s+2} \right), \quad k = 0.1, 0.5, 1.0, \text{ and } 2.0.$$



Bode sensitivity integral (r.h.p. poles and a r.h.p. zero)

$$\int_0^{\infty} \ln |S(j\omega)| w(z_0, \omega) d\omega = \pi \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z_0}{p_i - z_0} \right|$$

$$\text{As } p_i \rightarrow z_0 \quad \left| \frac{p_i + z_0}{p_i - z_0} \right| \rightarrow \infty$$

Interpolation conditions

Stable closed-loop $S(s)$ and $T(s)$

For RHP-poles, p_i , of $L(s)$,

$$T(p_i) = 1, \quad S(p_i) = 0$$

for RHP-zeros, z_i , of $L(s)$,

$$T(z_i) = 0, \quad S(z_i) = 1$$

Maximum modulus principle

Given a function, $f(s)$, analytic in the complex RHP (i.e. stable).

$$\text{Then, } \sup_{s \in \text{RHP}} |f(s)| = \sup_{s=j\omega} |f(s)|.$$

The maximum modulus is achieved on the $j\omega$ axis.

Maximum modulus principle

Given a function, $f(s)$, analytic in the complex RHP (i.e. stable).

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The maximum modulus is achieved on the $j\omega$ axis.

$$\|f(s)\|_{\mathcal{H}_\infty} = \sup_{\omega} |f(j\omega)| \geq |f(s)| \quad \text{for all } s \in \text{RHP}.$$

Weighted sensitivity peak

Suppose $G(s)$ has a RHP-zero, z .

Suppose we also have a sensitivity weight, $W_S(s)$.

Then for a closed-loop stable sensitivity function, $S(s)$,

$$\begin{aligned}\|W_S(s)S(s)\|_{\mathcal{H}_\infty} &\geq |W_S(z)S(z)| && \text{(via maximum modulus)} \\ &= |W_S(z)| && \text{(via interpolation conditions)}\end{aligned}$$

Performance requirements

$$\|W_S(s)S(s)\|_{\mathcal{H}_\infty} < 1 \quad \implies \quad |W_S(z)| < 1.$$

Weighted complementary sensitivity peak

Suppose $G(s)$ has a RHP-pole, p .

Suppose we also have a complementary sensitivity weight, $W_T(s)$.

Then for a closed-loop stable sensitivity function, $T(s)$,

$$\begin{aligned}\|W_T(s)T(s)\|_{\mathcal{H}_\infty} &\geq |W_T(p)T(p)| && \text{(via maximum modulus)} \\ &= |W_T(p)| && \text{(via interpolation conditions)}\end{aligned}$$

Noise performance (& robustness) requirements

$$\|W_T(s)T(s)\|_{\mathcal{H}_\infty} < 1 \quad \implies \quad |W_T(p)| < 1.$$

RHP poles and RHP zeros

Suppose $G(s)$ has N_z RHP zeros, z_j , and N_p RHP poles, p_i .

Then,

$$\|S(s)\|_{\mathcal{H}_\infty} \geq \max_j c_j, \quad c_j = \prod_{i=1}^{N_p} \frac{|z_j + \bar{p}_i|}{|z_j - p_i|} > 1$$

$$\|T(s)\|_{\mathcal{H}_\infty} \geq \max_i d_i, \quad d_i = \prod_{j=1}^{N_z} \frac{|\bar{z}_j + p_i|}{|z_j - p_i|} > 1$$

If the RHP pole and zero are close together these bounds are very large.

Closed-loop bandwidth with a RHP zero

Suppose $G(s)$ has a real RHP-zero, $z > 0$.

$$\|W_S(s)S(s)\|_{\mathcal{H}_\infty} \geq |W_S(z)| \implies |W_S(z)| < 1.$$

Say,

$$W_S(s) = \frac{s/M + \omega_B}{s + \omega_B A}$$

Substituting $s = z$ implies,

$$\omega_B(1 - A) < z \left(1 - \frac{1}{M}\right)$$

$$\text{For example: } A = 0, \quad M = 2 : \quad \omega_B < \frac{z}{2}$$

Closed-loop bandwidth with a RHP pole

Suppose $G(s)$ has a real RHP-pole, $p > 0$.

$$\|W_T(s)T(s)\|_{\mathcal{H}_\infty} \geq |W_T(p)| \implies |W_T(p)| < 1.$$

Say,

$$W_T(s) = \frac{s}{\omega_{BT}} + \frac{1}{M_T}$$

Substituting $s = p$ implies,

$$\omega_{BT} > p \frac{M_T}{M_T - 1}$$

For example: $M_T = 2$ implies that $\omega_{BT} > 2p$

Hierarchy

Difficult control problems (easier to harder):

- ▶ $G(s)$ stable and no r.h.p. zeros
(Bode integral = 0, max. control frequency limit)
- ▶ $G(s)$ stable and r.h.p. zero
(constrained Bode integral = 0, maximum bandwidth limit)
- ▶ $G(s)$ unstable, no r.h.p. zero
(Bode integral > 0, minimum bandwidth constraint)
- ▶ $G(s)$ unstable, r.h.p. zero and $p < z$,
(Bode integral $\gg 0$, min. and max. bandwidth constraints)
- ▶ $G(s)$ unstable, r.h.p. zero and $z < p$
(all hope of any performance is gone).

Notes and references

Skogestad & Postlethwaite (2nd Ed.)

“Perfect” control sections, 5.1 & 5.4

Bode sensitivity integrals: section 5.2

$\|S(s)\|_{\mathcal{H}_\infty}$ and $\|T(s)\|_{\mathcal{H}_\infty}$ bounds section 5.3

Bode sensitivity integrals

J. Freudenberg & D. Looze, “Right half plane poles and zeros and design trade-offs in feedback systems,” *IEEE Trans. Auto. Ctrl.*, **30**(6), pp. 555–565, 1985.