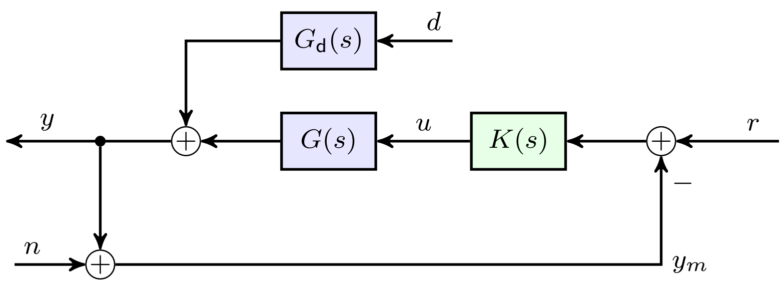
Control Systems 2

Lecture 3: Weighted sensitivity design

Roy Smith

Loopshaping



$$e \;=\; \underbrace{\frac{1}{1+L(s)}}_{S(s)} r \;-\; \underbrace{\frac{1}{1+L(s)}}_{S(s)} G_d(s) d \;+\; \underbrace{\frac{L(s)}{1+L(s)}}_{T(s)} n$$

Loopshaping

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In certain frequency ranges $|L(j\omega)|$.

$$|L(j\omega)| \gg 1 \quad \Longrightarrow \quad S(j\omega) \approx L(j\omega)^{-1} \quad T(j\omega) \approx 1$$

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Loopshaping

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In certain frequency ranges $|L(j\omega)|$.

$$|L(j\omega)| \gg 1 \quad \Longrightarrow \quad S(j\omega) \approx L(j\omega)^{-1} \quad T(j\omega) \approx 1$$

$$|L(j\omega)| \ll 1 \quad \Longrightarrow \quad S(j\omega) \approx 1; \quad T(j\omega) \approx L(j\omega)$$

In the crossover region the inferences about $S(j\omega)$ and $T(j\omega)$ from $|L(j\omega)|$ are limited.

$\mathcal{H}_\infty$ norm

The $\mathcal{H}_\infty$ norm is a measure of the "size" or "gain" of a system.

If $y(s) = G(s)u(s)$ (and stable) then,

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{u(s)\neq 0} \frac{\|y(s)\|_2}{\|u(s)\|_2}$$

$$= \sup_{u(s)\neq 0} \frac{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |y(j\omega)|^2 d\omega\right)^{1/2}}{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |u(j\omega)|^2 d\omega\right)^{1/2}}$$

$$= \max_{\omega} |G(j\omega)| \quad = \|G(s)\|_{\mathcal{H}_\infty} \quad \text{(alternative notation)}$$

$\mathcal{H}_\infty$ is the set of stable, $\mathcal{H}_\infty$-norm bounded transfer functions.

Weighted sensitivity specifications

Specifications that can be expressed with $S(j\omega)$.

1. Minimum bandwidth: ω_B.

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4. Shape of $S(j\omega)$ over selected frequency ranges.

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5. Maximum peak magnitude: $\|S(j\omega)\|_{\mathcal{H}_\infty} \leq M_S$.

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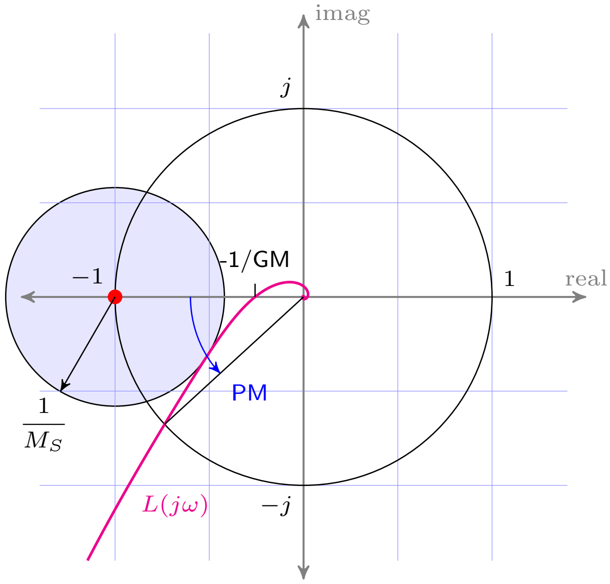
3. System type or maximum steady-state tracking error.
4. Shape of $S(j\omega)$ over selected frequency ranges.
5. Maximum peak magnitude: $\|S(j\omega)\|_{\mathcal{H}_\infty} \leq M_S$.

These will be specified by a frequency dependent upper bound on $|S(j\omega)|$.

Weighted sensitivity specifications

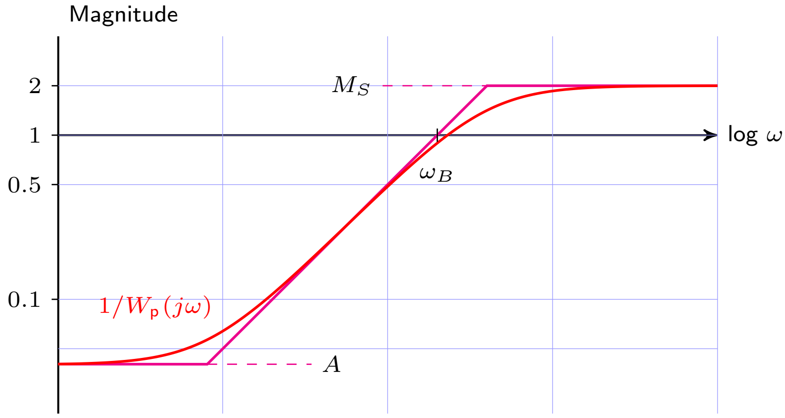
$$\|S(s)\|_{\mathcal{H}_\infty} \le M_S \quad \text{implies} \quad \text{GM} \ge \frac{M_S}{M_S - 1}, \quad \text{PM} \ge \frac{1}{M_S}\text{(rad)}$$



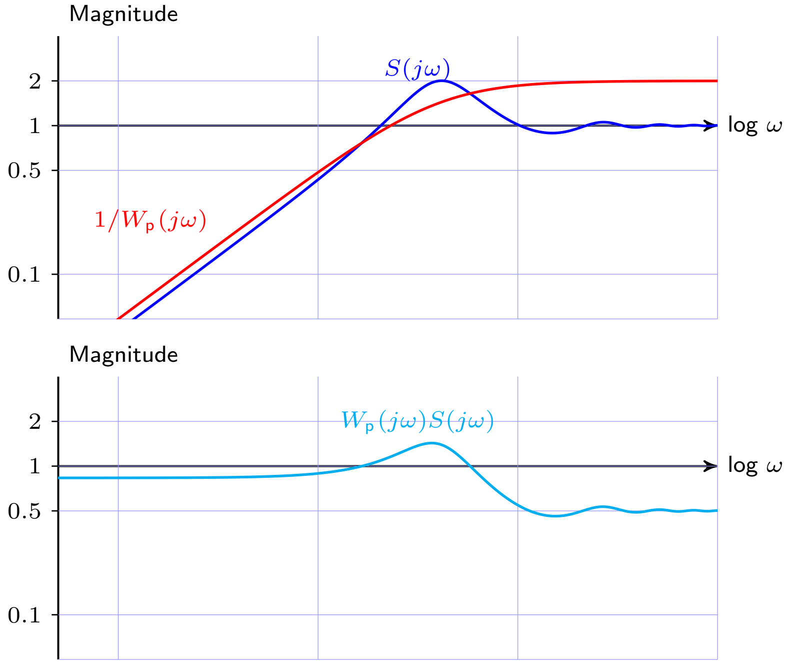
Weighted sensitivity specifications

$$|S(j\omega)| \le \left|\frac{1}{W_\mathsf{p}(j\omega)}\right| \quad \text{for all } \omega.$$

$$W_\mathsf{p}(s) = \frac{s/M_S + \omega_B}{s + w_B A}$$



Weighted sensitivity specifications



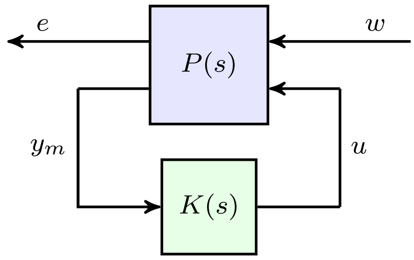
Weighted sensitivity specifications

$$|S(j\omega)| \leq \left| \frac{1}{W_{\mathsf{p}}(j\omega)} \right| \quad \text{for all } \omega$$

$$\iff \quad |W_{\mathsf{p}}(j\omega)S(j\omega)| \leq 1 \quad \text{for all } \omega$$

$$\iff \quad \|W_{\mathsf{p}}(s)S(s)\|_{\mathcal{H}_\infty} \leq 1.$$

General weighted design problem



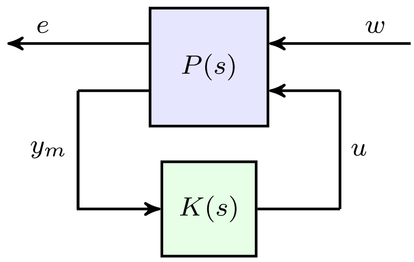
e = performance outputs $\qquad$ w = exogenous inputs

y_m = controller measurements $\qquad$ u = control actuation

Closed-loop transfer function: $e(s) = N(s)w(s)$,

$$N(s) = \mathcal{F}_l\left(P(s), K(s)\right) = P_{11} + P_{12}\left(I - KP_{22}\right)^{-1}KP_{21} = P_{11} + P_{12}K\left(I - P_{22}K\right)^{-1}P_{21}$$

General weighted design problem



Closed-loop transfer function: $e(s) = N(s)w(s)$,

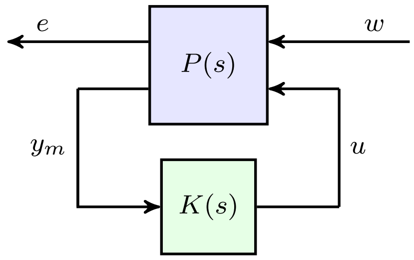
$$N(s) = P_{11} + P_{12}\left(I - KP_{22}\right)^{-1}KP_{21}$$

Design problem:

Find $K(s)$ such that:

- $N(s)$ is stable (closed-loop stability)
- $\|N(s)\|_{\mathcal{H}_\infty} \leq 1$

General weighted design problem



Closed-loop transfer function: $e(s) = N(s)w(s)$,

$$N(s) = P_{11} + P_{12}\left(I - KP_{22}\right)^{-1} KP_{21}$$

Design problem:

$$\underset{K(s)\text{ stabilizing}}{\text{minimize}} \; \|\mathcal{F}_l\left(P(s), K(s)\right)\|_{\mathcal{H}_\infty}$$

Mixed sensitivity design

We typically have additional performance objectives:

Weighted objectives:

- $\|W_{\mathsf{p}}(s)S(s)\|_{\mathcal{H}_\infty} \leq 1$ (weighted sensitivity; tracking error)
- $\|W_u(s)K(s)S(s)\|_{\mathcal{H}_\infty} \leq 1$ (actuation limits)
- $\|W_{\mathsf{m}}(s)T(s)\|_{\mathcal{H}_\infty} \leq 1$ (complementary sensitivity, robustness)

Vector valued output:

$$N(s) = \begin{bmatrix} W_{\mathsf{p}}(s)S(s) \\ W_u(s)K(s)S(s) \\ W_{\mathsf{m}}(s)T(s) \end{bmatrix}$$

Mixed sensitivity design

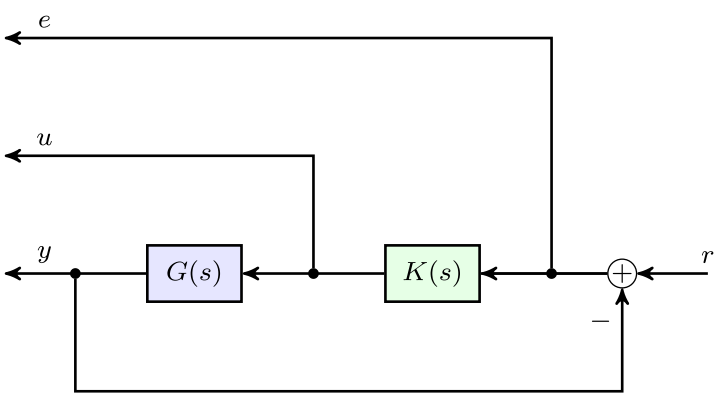
Vector valued output:

$$N(s) = \begin{bmatrix} W_{\mathsf{p}}(s)S(s) \\ W_u(s)K(s)S(s) \\ W_{\mathsf{m}}(s)T(s) \end{bmatrix}$$

Matrix objectives (potentially w and e) are both vectors.

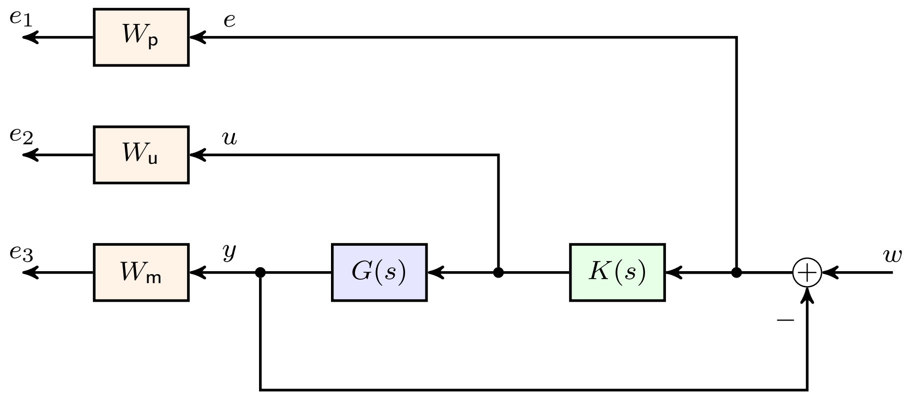
$$\begin{aligned} \|N(s)\|_{\mathcal{H}_\infty} &:= \sup_\omega \overline{\sigma}\left(N(j\omega)\right) \\ &= \sup_\omega \sqrt{|W_{\mathsf{p}}S|^2 + |W_uKS|^2 + |W_{\mathsf{m}}T|^2} \end{aligned}$$ (single-input case)

Interconnection structure



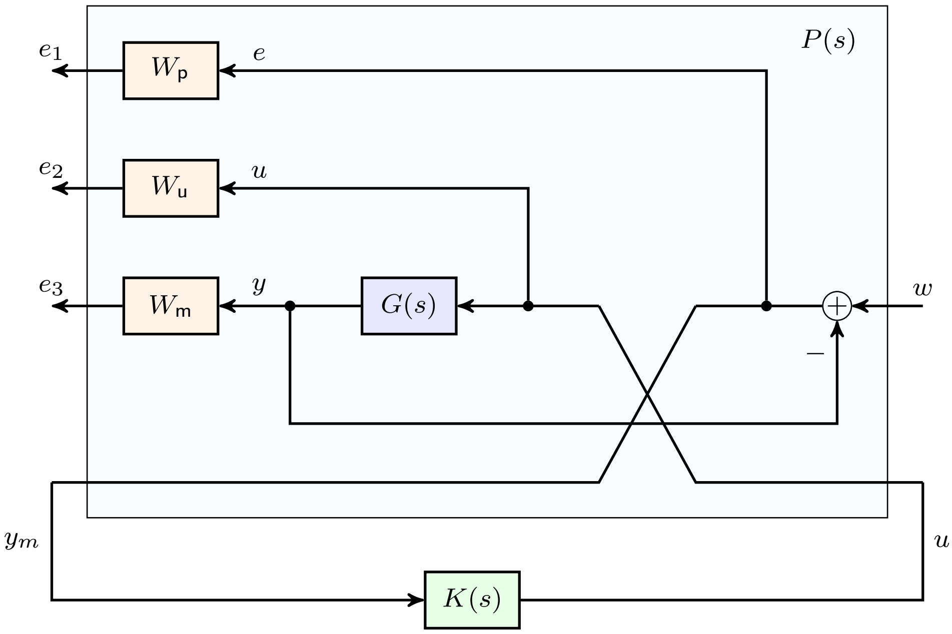
$$N(s) = \begin{bmatrix} W_{\mathsf{p}}(s)S(s) \\ W_{\mathsf{u}}(s)K(s)S(s) \\ W_{\mathsf{m}}(s)T(s) \end{bmatrix}$$

Interconnection structure



$$N(s) = \begin{bmatrix} W_p(s)S(s) \\ W_u(s)K(s)S(s) \\ W_m(s)T(s) \end{bmatrix}$$

Interconnection structure



Example: mixed sensitivity design

$$G(s) = \frac{200}{(10s+1)} \frac{1}{(0.05s+1)^2}, \qquad G_{\mathsf{d}}(s) = \frac{100}{10s+1}$$

$$N(s) = \begin{bmatrix} W_{\mathsf{p}}(s)S(s) \\ W_u(s)K(s)S(s) \end{bmatrix}$$

Sensitivity performance weights (two designs):

Design 1

Sensitivity:	$W_{\mathsf{p}_1} = \frac{s/M_{S_1}+\omega_{B_1}}{(s+\omega_{B_1}A_1)}$
Bandwidth:	$\omega_{B_1} = 7.5$ [rad/sec]
$\|S\|_{\mathcal{H}_\infty}$	$M_{S_1} = 2.4$
DC error:	$A_1 = 0.00016$
Actuator:	$W_{\mathsf{u}_1} = 0.625$

Design 2

Sensitivity:	$W_{\mathsf{p}_2} = \frac{(s/\sqrt{M_{S_2}}+\omega_{B_2})^2}{(s+\omega_{B_2}\sqrt{A_2})^2}$
Bandwidth:	$\omega_{B_2} = 7.24$ [rad/sec]
$\|S\|_{\mathcal{H}_\infty}$	$M_{S_2} = 4.13$
DC error:	$A_2 = 0.00028$
Actuator:	$W_{\mathsf{u}_2} = 0.36$

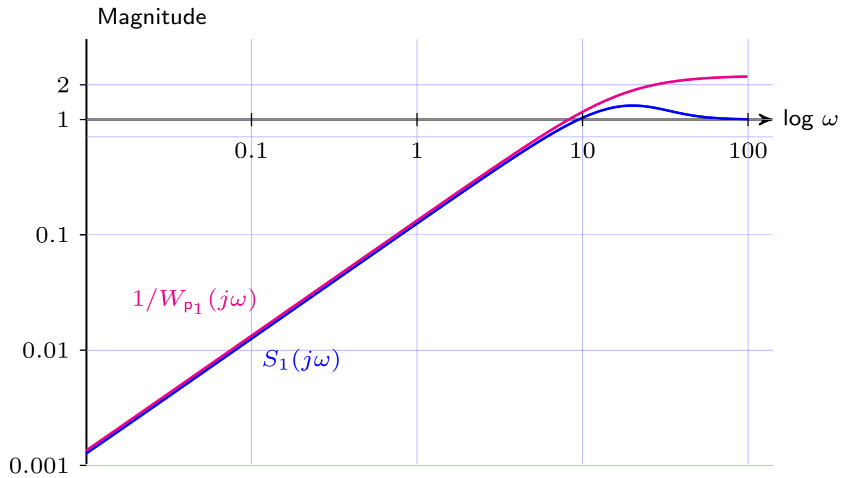
Example: mixed sensitivity design

Design 1

$$W_{\mathsf{p}_1} = \frac{s/M_{S_1}+\omega_{B_1}}{s+\omega_{B_1}A_1}, \qquad \omega_{B_1} = 7.5, \quad M_{S_1} = 2.4, \quad A_1 = 1.6\times 10^{-4}$$

$$W_{\mathsf{u}_1} = 0.625$$

Sensitivity bounds



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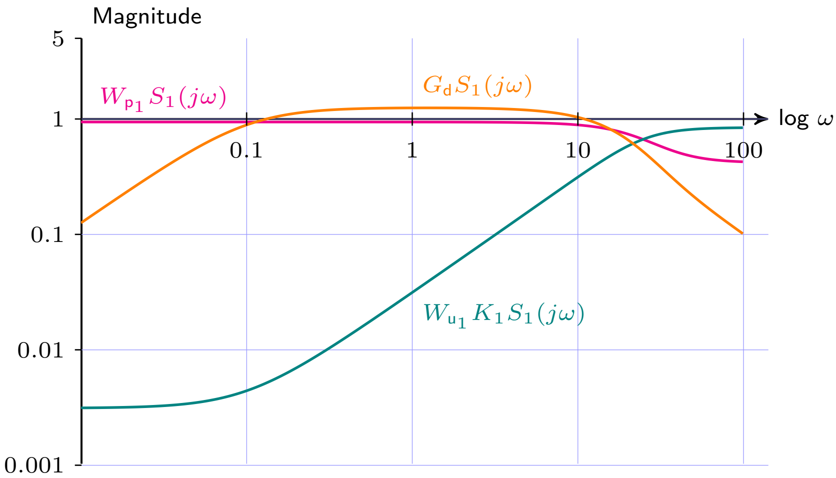
Weighted sensitivity specifications



Example: mixed sensitivity design

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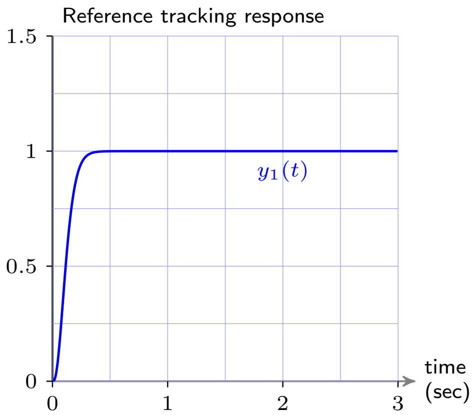
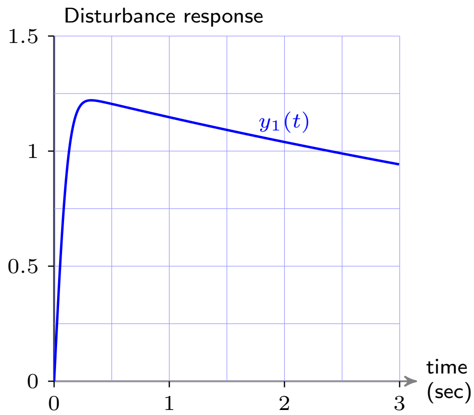
Time-domain responses





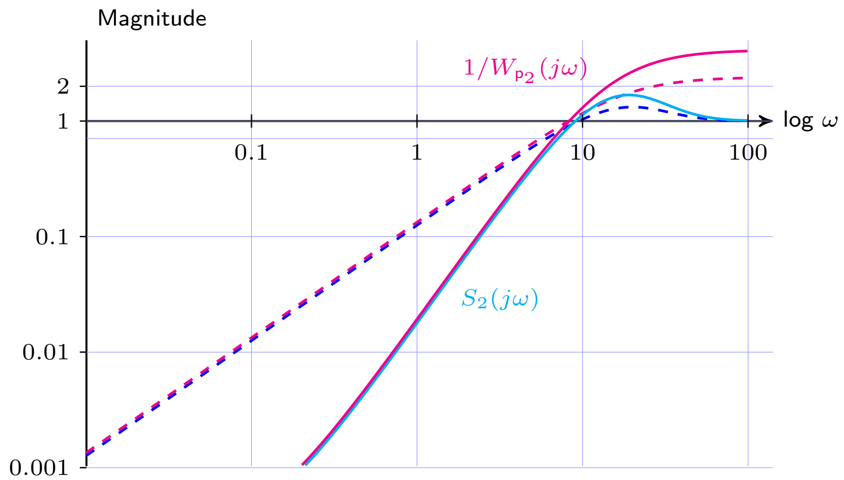
Example: mixed sensitivity design

Design 2

$$W_{p_2} = \frac{(s/\sqrt{M_{S_2}} + \omega_{B_2})^2}{(s + \omega_{B_2}\sqrt{A_2})^2}, \qquad \omega_{B_2} = 7.24, \quad M_{S_2} = 4.13, \quad A_2 = 2.8 \times 10^{-4}$$

$$W_{u_2} = 0.364$$

Sensitivity bounds



Example: mixed sensitivity design

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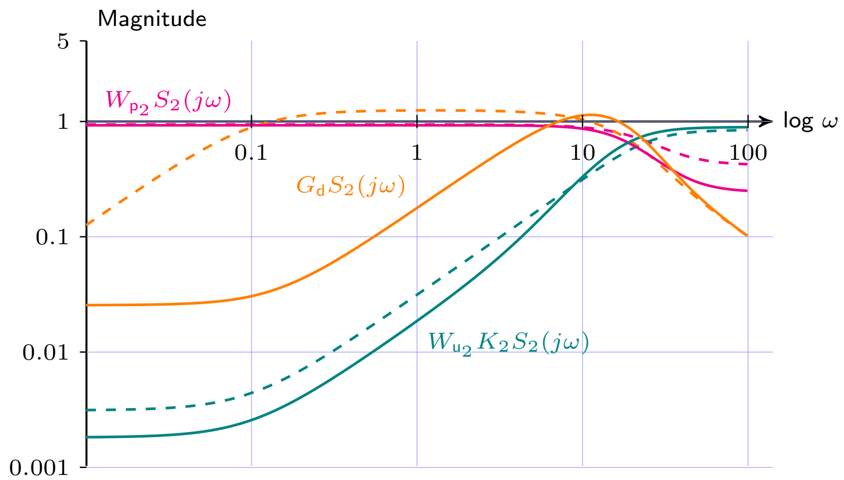
Weighted sensitivity specifications



Example: mixed sensitivity design

Design 2

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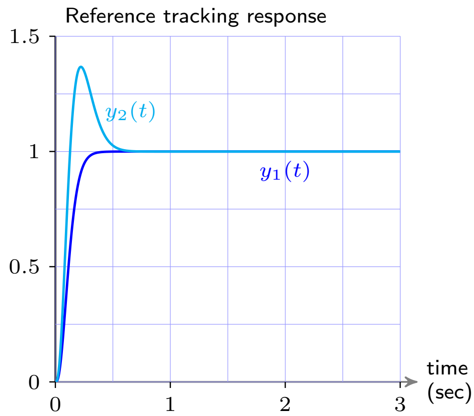
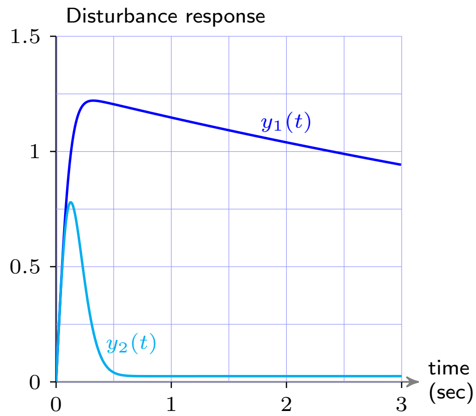
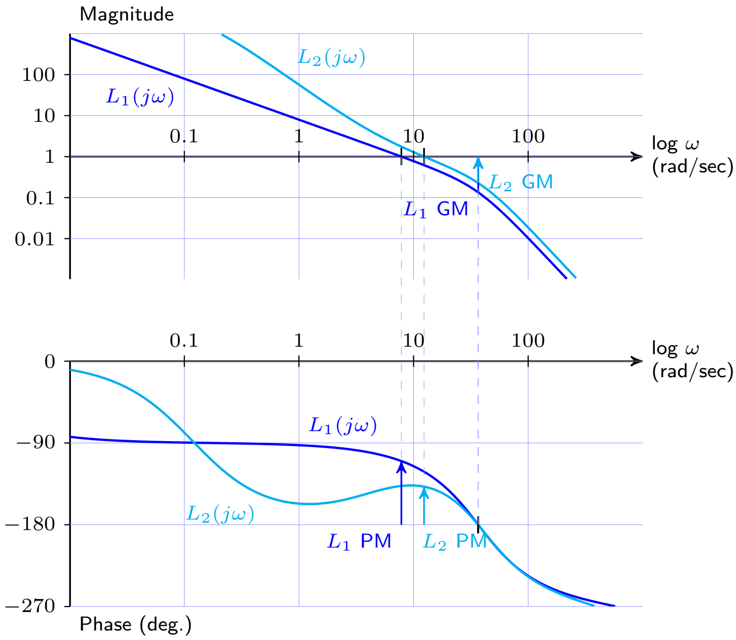




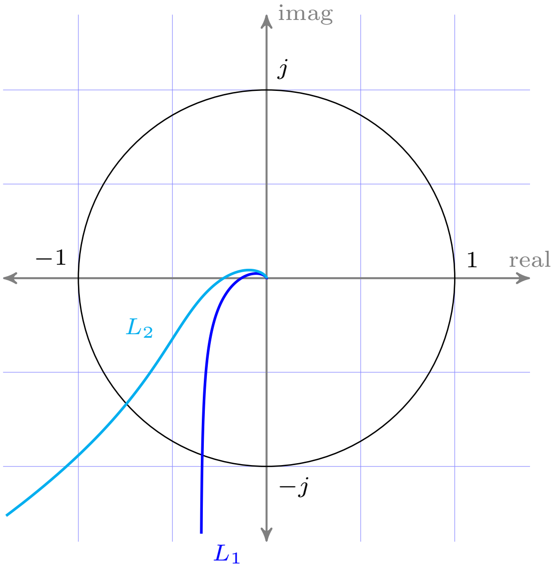
Example: mixed sensitivity design

Loopshape comparison



Example: mixed sensitivity design

Nyquist comparison



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$$\omega_{B_1} = 7.5$$
$$A_1 = 0.00016$$
$$M_{S_1} = 2.4$$

Design 2

$$W_{p_2} = \frac{(s/\sqrt{M_{S_2}} + \omega_{B_2})^2}{(s + \omega_{B_2}\sqrt{A_2})^2}$$
$$\omega_{B_2} = 7.24$$
$$A_2 = 0.00028$$
$$M_{S_2} = 4.13$$

Mixed sensitivity design

Exercise

How would you change this design to:

- give a 2-degree-of-freeedom controller; and
- weight the disturbance response differently from the reference tracking response?

Notes and references

Skogestad & Postlethwaite (2nd Ed.)

Weighted sensitivity: section 2.8

General control formulation: section 3.8