Chapter

5 Glacier flow

5.1 Glacier flow theories

> "Wie ist es möglich, wie kommt es zustande, dass dieser scheinbar starre Körper, das Eis, thalabwärts fliesst?" (Albert Heim, 1895; p. 290)

Which processes are responsible for glacier flow was a highly debated topic until about 60 years ago. First written explanations stem from 1705 (by J. J. Scheuchzer). Many contradicting theories tried to explain how a glacier moves (thorough discussion in Heim (1895), pp. 293 – 337). According to these theories, a glacier moves because of ...

- Dilatation theories
  - water freezing in crevasses and veins within the glacier
  - growing ice grains push glacier downward
  - dilatation and contraction due to temperature variations
  - sun light penetrates the ice and fluidizes it for a short moment
- Gravitation theories
  - plasticity, fluidity without fracturing
  - plasticity, fluidity with fracturing and regelation
  - sliding motion over the bedrock

It is interesting to notice that for us most of the above processes sound amusing, even if all of them (and more) are real. Now that we have a good theory with predictive power (although far from perfect) we know which processes dominate, and which ones are negligible or just boundary effects (e.g. penetration of sunlight).

To understand the foundation of modern glacier flow theories, we first look at the structure and deformation of individual ice crystals and small crystal assemblies.

5.2 Ice physics

Solid ice comes in twelve (!) different phases with fundamentally different crystal structure, and two amorphous states, depending on temperature, pressure and crystallization history (Fig. 5.1). Under usual atmospheric conditions only **ice Ih** (phase I in a hexagonal lattice) is formed. There is evidence that ice Ic with a cubic lattice is formed in very cold, high altitude clouds.

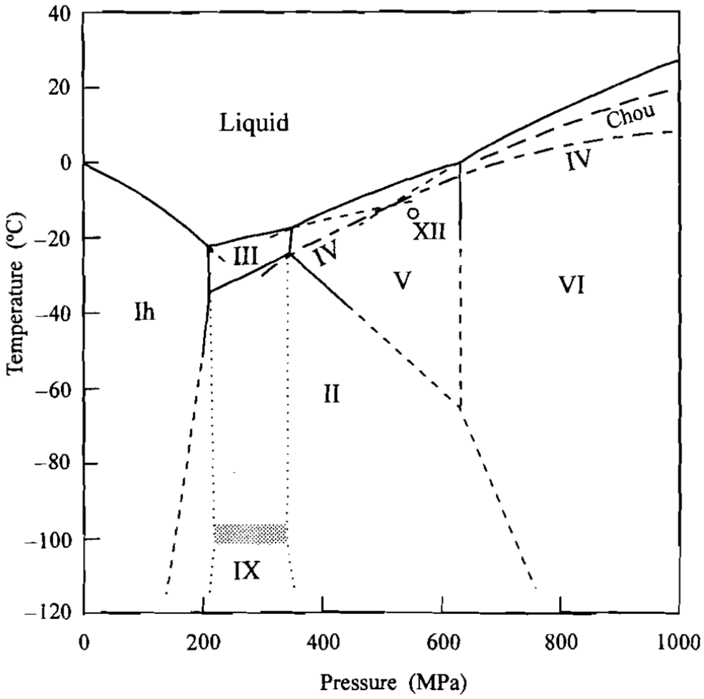

Figure 5.1: *Phase diagram of the solid phases of ice (from Petrenko and Withworth, 1999). Notice that maximum stresses in polar ice caps are* 40 MPa*, such that on the Earth surface only ice Ih occurs.*

Glacier ice is a polycrystalline material that consists of many crystals of different sizes and orientation. A single ice crystal mainly deforms by gliding on its basal planes (which is described by the theory of dislocations). Typical crystal sizes in glaciers range from the sub-millimeter scale in freshly formed or highly deformed ice to decimeters (tens of centimeters) in very old and slow moving ice. If not affected by deviatoric stresses, the cross sectional area D of ice grains increases linearly with time $D - D_0 = kt$, where the growth factor k is mainly temperature dependent. Laboratory experiments of natural and artificial ice at $-2°$ C gave $k = 85 - 260\,\mathrm{mm^2\,a^{-1}}$ (Azuma and Higashi, 1983).

Polycrystalline ice deforms in creeping flow because of processes within and between the grains. The most important among them are dislocation climb (within grains), grain boundary migration, grain rotation, and dynamic recrystallization (formation of new grains). Figure 5.2 gives an impression of the behavior of an assembly of ice crystals under different applied forces.

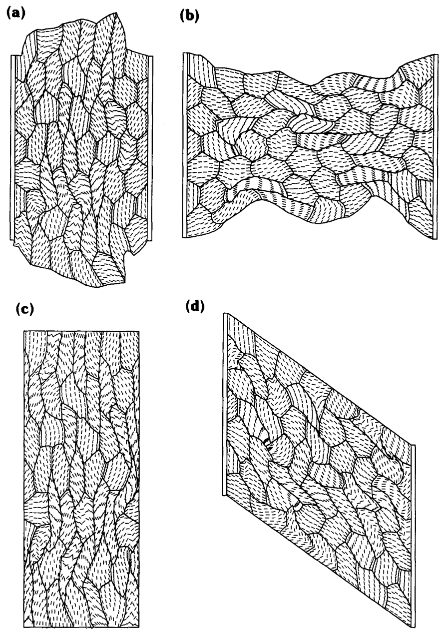

Figure 5.2: *Model of a deforming aggregate of polycrystalline ice. The lines within crystals indicate c-axis orientations. (a) axial shortening by* 29 % *(b) axial extension by* 33 % *(c) pure shear by* 38 % *(d) simple shearing* $\gamma = 0.72$ *(from Zhang et al., 1994)*

If stress is applied to a block of glacier ice, the ice immediately deforms *elastically*. Permanent deformation – termed *creep* or *viscous flow* – then begins and continues as long as stress is applied. If the stress is removed, only the elastic deformation can be recovered. Viscous flow (creep) is a dissipative process that transforms mechanical energy into heat which cannot be turned back into mechanical energy. Depending on the initial crystal structure, measured strain rates vary in laboratory experiments. After $1 - 3\,\%$ of strain the crystal is completely recrystallized, and initial anisotropies are no more discernible. Figure 5.3 shows typical creep curves measured in laboratory experiments. Due to the long time spans and high strains, only secondary and tertiary creep are usually considered for glacier flow.

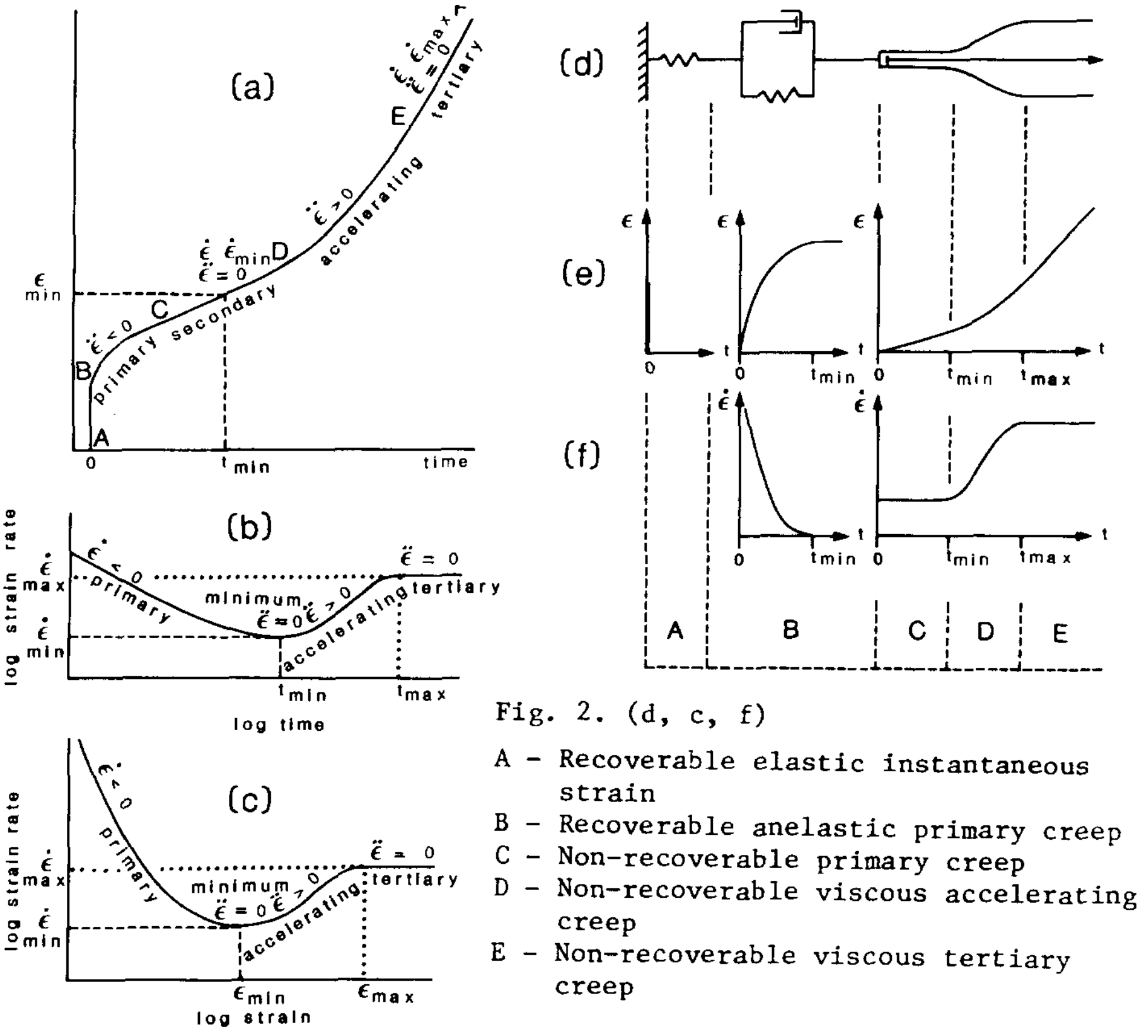

Figure 5.3: *Ice creep curve and rheological model. (a) An idealised creep curve for ice shows the various stages of strain e versus time t as follows: A-B: the initial impulsive elastic strain, B-C, C-D, D-E, E-F the primary, secondary, accelerating and tertiary stages. (b) The corresponding log strain rate versus log time plot is shown indicating the minimum and "steady-state" tertiary strain rates. (c) The corresponding log strain rate (* $\log \dot{\epsilon}$ *) versus log strain (* $\log \epsilon$ *) shows the same features. (d) A simple rheological model is indicated which results in the various stages and components as described above and indicated by the corresponding strains in (e) and strain rates in (f). (From Budd and Jacka, 1989).*

5.3 Flow relation for polycrystalline ice

The most widely used flow relation for glacier ice is (Glen, 1952; Nye, 1957)

$$\dot{\varepsilon}_{ij} = A\tau^{n-1}\sigma_{ij}^{(d)}. \tag{5.1}$$

with $n \sim 3$. The rate factor $A = A(T)$ depends on temperature (Fig. 5.4; Table A1) and other parameters like water content, impurity content and crystal size. The quantity τ is the second invariant of the deviatoric stress tensor in Equation (4.14). Several properties of Equation (5.1) are noteworthy:

- Elastic effects are neglected. This is reasonable if processes on the time scale of days and longer are considered.
- Stress and strain rate are collinear, i.e. a shear stress leads to shearing strain rate, a compressive stress to a compression strain rate, and so on.
- Only deviatoric stresses lead to deformation rates, isotropic pressure alone cannot induce deformation. Ice is an *incompressible* material (no volume change, except for elastic compression). This is expressed as

$$\dot{\varepsilon}_{ii} = 0 \qquad \Longleftrightarrow \qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

- A *Newtonian viscous fluid*, like water, is characterized by the *viscosity* η

$$\dot{\varepsilon}_{ij} = \frac{1}{2\eta}\sigma_{ij}^{(d)}. \tag{5.2}$$

  By comparison with Equation (5.1) we find that viscosity of glacier ice is

$$\eta = \frac{1}{2A\tau^{n-1}}.$$

- Polycrystalline glacier ice is a *viscous fluid* with a **stress dependent viscosity** (or, equivalently, a strain rate dependent viscosity). Such a material is called a *non-Newtonian fluid*, or more specifically a *power-law fluid.*
- Polycrystalline glacier ice is treated as an *isotropic fluid.* No preferred direction (due to crystal orientation fabric) appears in the flow relation. This is a crude approximation to reality, since glacier ice usually is anisotropic, although to varying degrees.

Many alternative flow relations have been proposed that take into account the compressibility of firn at low density, the anisotropic nature of ice, microcracks and damaged ice, the water content, impurities and different grain sizes. Glen's flow law is still widely used because of its simplicity and ability to approximately describe most processes relevant to glacier dynamics at large scale.

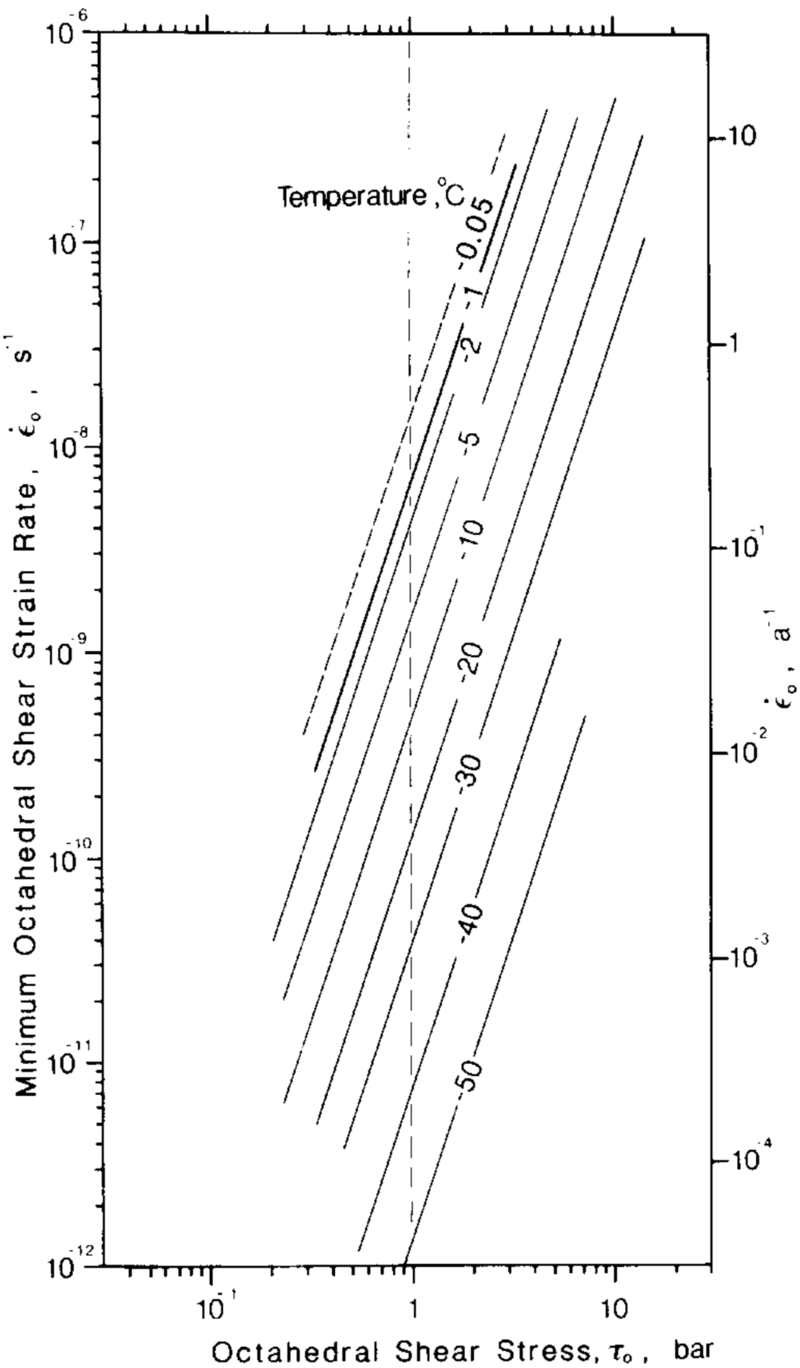

Figure 5.4: *Composite flow law for minimum strain rates from shear and compression tests. The minimum octahedral shear strain rate $\dot{\epsilon}_0$ is shown as a function of octahedral shear stress τ_0 on log-log coordinates for the temperatures -0.05, -1, -2, -5, -10, -20, -30, -40, -50 $^\circ$C over the range of τ_0 from 0.2-10 bars. The slopes of the lines correspond to a power law index of about $n = 3$. (From Budd and Jacka, 1989).*

Inversion of the flow relation

The flow relation of Equation (5.1) can be inverted so that stresses are expressed in terms of strain rates. Multiplying Equation (5.1) with itself gives

$$\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij} = A^2\tau^{2(n-1)}\sigma_{ij}^{(d)}\sigma_{ij}^{(d)} \qquad \text{(multiply by } \frac{1}{2}\text{)}$$

$$\underbrace{\frac{1}{2}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}}_{\dot{\epsilon}^2} = A^2\tau^{2(n-1)}\underbrace{\frac{1}{2}\sigma_{ij}^{(d)}\sigma_{ij}^{(d)}}_{\tau^2}$$

where we have used the definition for the *effective strain rate* $\dot{\epsilon} = \dot{\varepsilon}_e$, in analogy to the *effective shear stress* $\tau = \sigma_e$

$$\dot{\epsilon} = \sqrt{\frac{1}{2}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}}\,. \tag{5.3}$$

This leads to a relation between tensor invariants

$$\dot{\epsilon} = A\tau^n\,. \tag{5.4}$$

Coincidentally this is also the equation to describe simple shear, the most important part of ice deformation in glaciers

$$\dot{\epsilon}_{xz} = A\sigma_{xz}^{(d)n}\,. \tag{5.5}$$

Now we can invert the flow relation Equation (5.1)

$$\begin{aligned}
\sigma_{ij}^{(d)} &= A^{-1}\tau^{1-n}\,\dot{\varepsilon}_{ij}\\
\sigma_{ij}^{(d)} &= A^{-1}A^{\frac{n-1}{n}}\,\dot{\epsilon}^{-\frac{n-1}{n}}\,\dot{\varepsilon}_{ij}\\
\sigma_{ij}^{(d)} &= A^{-\frac{1}{n}}\,\dot{\epsilon}^{-\frac{n-1}{n}}\,\dot{\varepsilon}_{ij}\,.
\end{aligned} \tag{5.6}$$

The above relation allows us to calculate the stress state if the strain rates are known (from measurements). Notice that only deviatoric stresses can be calculated. The mean stress (pressure) cannot be determined because of the incompressibility of the ice. Comparing Equation (5.6) with (5.2) we see that the shear viscosity is

$$\eta = \frac{1}{2}A^{-\frac{1}{n}}\,\dot{\epsilon}^{-\frac{n-1}{n}}\,. \tag{5.7}$$

Polycrystalline ice is a *strain rate softening* material: viscosity decreases as the strain rate increases.

Notice that the viscosity given in Equation (5.7) becomes infinite at very low strain rates, which of course is unphysical. One way to alleviate that problem is to add a small quantity η_o to obtain a *finite viscosity*

$$\eta^{-1} = \left(\frac{1}{2}A^{-\frac{1}{n}}\,\dot{\epsilon}^{-\frac{n-1}{n}}\right)^{-1} + \eta_0^{-1}. \tag{5.8}$$

5.4 Simple stress states

To see what Glen flow law of Equation (5.1) describes, we investigate some simple, yet important stress states imposed on small samples of ice, e.g. in the laboratory. Only surface forces are applied, and body forces, such as gravity, are neglected.

a) Simple shear

$$\dot{\varepsilon}_{xz} = A(\sigma_{xz}^{(d)})^3 = A\sigma_{xz}^3 \tag{5.9}$$

This stress regime applies near the base of a glacier.

b) Unconfined uniaxial compression along the vertical z-axis

$$\begin{aligned}
\sigma_{xx} &= \sigma_{yy} = 0 \\
\sigma_{zz}^{(d)} &= \frac{2}{3}\sigma_{zz}; \qquad \sigma_{xx}^{(d)} = \sigma_{yy}^{(d)} = -\frac{1}{3}\sigma_{zz} \\
\dot{\varepsilon}_{xx} &= \dot{\varepsilon}_{yy} = -\frac{1}{2}\dot{\varepsilon}_{zz} = -\frac{1}{9}A\sigma_{zz}^3 \\
\dot{\varepsilon}_{zz} &= \frac{2}{9}A\sigma_{zz}^3
\end{aligned} \tag{5.10}$$

This stress system is easy to investigate in laboratory experiments, and also applies in the near-surface layers of an ice sheet. The deformation rate is only 22 % of the deformation rate at a shear stress of equal magnitude (Eq. 5.9).

c) Uniaxial compression confined in the y-direction

$$\begin{aligned}
\sigma_{xx} &= 0; \qquad \dot{\varepsilon}_{yy} = 0; \qquad \dot{\varepsilon}_{xx} = -\dot{\varepsilon}_{zz} \\
\sigma_{yy}^{(d)} &= \frac{1}{3}\left(2\sigma_{yy} - \sigma_{zz}\right) = 0; \qquad \sigma_{yy} = \frac{1}{2}\sigma_{zz} \\
\sigma_{xx}^{(d)} &= -\sigma_{zz}^{(d)} = -\frac{1}{3}\left(\sigma_{yy} + \sigma_{zz}\right) = -\frac{1}{2}\sigma_{zz} \\
\dot{\varepsilon}_{zz} &= \frac{1}{8}A\sigma_{zz}^3
\end{aligned} \tag{5.11}$$

This stress system applies in the near-surface layers of a valley glacier and in an ice shelf occupying a bay.

d) Shear combined with unconfined uniaxial compression

$$
\begin{aligned}
\sigma_{xx} &= \sigma_{yy} = \sigma_{xy} = \sigma_{yz} = 0 \\
\sigma_{zz}^{(d)} &= \frac{2}{3}\sigma_{zz} = -2\sigma_{xx}^{(d)} = -2\sigma_{yy}^{(d)} \\
\tau^2 &= \frac{1}{3}\sigma_{zz}^2 + \sigma_{xz}^2 \\
\dot{\varepsilon}_{zz} &= -2\dot{\varepsilon}_{xx} = -2\dot{\varepsilon}_{yy} = \frac{2}{3}A\tau^2\sigma_{zz} \\
\dot{\varepsilon}_{xz} &= A\tau^2\sigma_{xz}
\end{aligned} \tag{5.12}
$$

This stress configuration applies at many places in ice sheets.

5.5 Field equations

To calculate velocities in a glacier we have to solve *field equations*. For a mechanical problem (e.g. glacier flow) we need the *continuity of mass* and the *force balance equations*. The **mass continuity** equation for a compressible material of density ρ is (in different notations)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{5.13a}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{5.13b}$$

If the density is homogeneous ($\frac{\partial \rho}{\partial x_i} = 0$) and constant (incompressible material $\frac{\partial \rho}{\partial t} = 0$) we get, in different, equivalent notations

$$\mathrm{tr}\, \dot{\boldsymbol{\varepsilon}} = \dot{\varepsilon}_{ii} = 0 \tag{5.14a}$$

$$\nabla \cdot \mathbf{v} = v_{i,i} = 0 \tag{5.14b}$$

$$\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz} = 0 \tag{5.14c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.14d}$$

The **force balance** equation describes that all forces acting on a volume of ice, including the body force $\mathbf{b} = \rho\,\mathbf{g}$ (where $\mathbf{g}$ is gravity), need to be balanced by forces acting on the sides of the volume. In compact tensor notation they read

$$\nabla \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}\,, \tag{5.15a}$$

The same equations rewritten in index notation (summation convention)

$$\sigma_{ij,j} + b_i = \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \tag{5.15b}$$

and in full, unabridged notation

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}+\rho g_x=0\\ \frac{\partial \sigma_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yz}}{\partial z}+\rho g_y=0\\ \frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}+\rho g_z=0\end{aligned} \qquad (5.15c)$$

These three equations describe how the body forces and boundary stresses are balanced by the stress gradients throughout the body.

Recipe

A recipe to calculate the flow velocities from given stresses, strain rates, symmetry conditions and boundary conditions

1. determine all components of the stress tensor σ_{ij} exploiting the symmetries, and using the flow law (5.1) or its inverse (5.6)
2. calculate the mean stress σ_m
3. calculate the deviatoric stress tensor $\sigma_{ij}^{(d)}$
4. calculate the effective shear stress (second invariant) τ
5. calculate the strain rates $\dot{\varepsilon}_{ij}$ from τ and $\sigma_{ij}^{(d)}$ using the flow law
6. integrate the strain rates to obtain velocities
7. insert boundary conditions

5.6 Parallel sided slab

Now we are in a position to calculate the velocity of a slab of ice resting on inclined bedrock. We assume that the inclination angle of surface α and bedrock β are the same. For simplicity we choose a coordinate system K that is inclined, i.e. the x-Axis is along the bedrock (Figure 5.5).

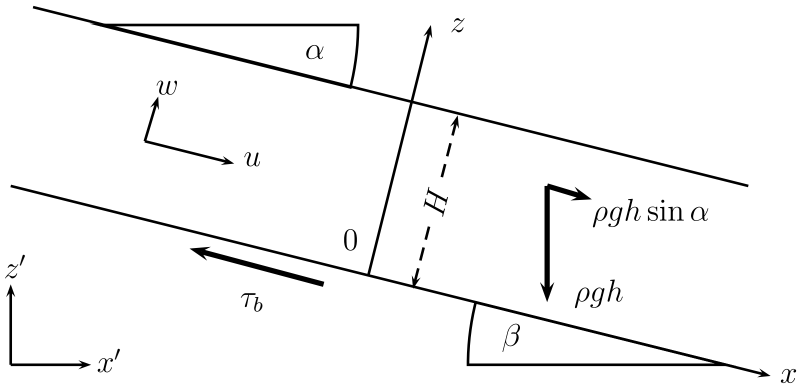

Figure 5.5: *Inclined coordinate system for a parallel sided slab.*

Body force In any coordinate system the body force (gravity) is $\mathbf{b} = \rho\mathbf{g}$. In the untilted coordinate system K' the body force is vertical along the $\hat{\mathbf{e}}'_3$ direction

$$\mathbf{b} = -\rho g\, \hat{\mathbf{e}}'_3\,. \tag{5.16}$$

The rotation matrix describes the transformation from K' to K (Appendix C.1)

$$[a_{ij}] = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \tag{5.17}$$

and therefore

$$b_i = a_{ij} b'_j$$

with the components

$$\begin{aligned} b_1 &= a_{11}b'_1 + a_{12}b'_2 + a_{13}b'_3 \\ &= \cos\alpha \cdot 0 + 0 \cdot 0 - \sin\alpha(-g) \\ &= \rho g \sin\alpha \\ b_2 &= 0 \\ b_3 &= -\rho g \cos\alpha\,. \end{aligned}$$

Symmetry The problem has translational symmetry in the x and y direction. It follows that none of the quantities can change in these directions

$$\frac{\partial\,(\cdot)}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial\,(\cdot)}{\partial y} = 0\,. \tag{5.18}$$

Furthermore no deformation takes place in the y direction, i.e. all deformation happens in the x, z plane. This is called *plane strain* and leads to the constraints

$$\dot{\varepsilon}_{yx} = \dot{\varepsilon}_{yy} = \dot{\varepsilon}_{yz} = 0. \tag{5.19}$$

Boundary conditions The glacier surface with the face normal $\hat{\mathbf{n}}$ is traction free

$$\mathbf{\Sigma}(\hat{\mathbf{n}}) \stackrel{!}{=} \mathbf{0} \iff \boldsymbol{\sigma}\hat{\mathbf{n}} \stackrel{!}{=} \mathbf{0} \iff \boldsymbol{\sigma}\begin{pmatrix}0\\0\\1\end{pmatrix} \stackrel{!}{=} \mathbf{0} \iff \begin{pmatrix}\sigma_{xz}\\\sigma_{yz}\\\sigma_{zz}\end{pmatrix} \stackrel{!}{=} \mathbf{0}\,. \tag{5.20}$$

The boundary conditions at the glacier base are $v_x = u_b$, $v_y = 0$ and $v_z = 0$.

Solution of the system We now insert all of the above terms into the field equations (5.14) and (5.15). The mass continuity equation (5.14d) together with (5.18) leads to

$$0 + 0 + \frac{\partial v_z}{\partial z} = 0. \tag{5.21}$$

The velocity component v_z is constant. Together with the boundary condition at the base ($v_z = 0$) we conclude that $v_z = 0$ everywhere.

Most terms in the momentum balance equation (5.15c) are zero, and therefore

$$\frac{\partial \sigma_{xz}}{\partial z} = -\rho g \sin\alpha\,, \tag{5.22a}$$

$$\frac{\partial \sigma_{yz}}{\partial z} = 0\,, \tag{5.22b}$$

$$\frac{\partial \sigma_{zz}}{\partial z} = \rho g \cos\alpha\,. \tag{5.22c}$$

Integration of Equation (5.22a) and (5.22c) with respect to z leads to

$$\begin{aligned}\sigma_{xz} &= -\rho g z \sin\alpha + c_1,\\ \sigma_{zz} &= \rho g z \cos\alpha + c_2.\end{aligned}$$

The integration constants c_1 and c_2 can be determined with the traction boundary conditions (5.20) at the surface ($z_s = H$) and lead to

$$\sigma_{xz}(z) = \rho g (H - z) \sin\alpha, \tag{5.23a}$$

$$\sigma_{zz}(z) = -\rho g (H - z) \cos\alpha. \tag{5.23b}$$

We see that the stresses vary linearly with depth.

To calculate the deformation rates we exploit Glen's Flow Law (5.1), and make use of the fact that the strain rate components are directly related to the deviatoric stress components. Obviously we need to calculate the deviatoric stress tensor $\boldsymbol{\sigma}^{(d)}$. Because of the plain strain condition (Eq. 5.19) and the flow law, the deviatoric stresses in y-direction vanish $\sigma_{iy}^{(d)} = 0$. By definition $\sigma_{yy}^{(d)} = \sigma_{yy} - \sigma_m$ so that $\sigma_{yy} = \sigma_m$, where use has been made of the definition of the mean stress $\sigma_m := \frac{1}{3}\sigma_{ii} = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$.

Using again Glen's flow law we also see that (Eq. 5.18a)

$$\dot{\varepsilon}_{xx} = \frac{\partial v_x}{\partial x} \overset{!}{=} 0 \qquad \text{leads to} \qquad \sigma_{xx}^{(d)} = 0 \qquad \text{and} \quad \sigma_{xx} = \sigma_m.$$

Therefore all diagonal components of the stress tensor are equal to the mean stress $\sigma_m = \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -\rho g(H - z)\cos\alpha$ (using Eq. 5.23b). The effective stress τ can now be calculated with the deviatoric stress components determined above

$$\begin{aligned} \sigma_{xx}^{(d)} &= \sigma_{yy}^{(d)} = \sigma_{zz}^{(d)} = 0, \\ \sigma_{xy}^{(d)} &= \sigma_{yz}^{(d)} = 0, \\ \sigma_{xz}^{(d)} &= \rho g(H - z)\sin\alpha, \end{aligned}$$

so that

$$\tau^2 = \frac{1}{2}\sigma_{ij}^{(d)}\sigma_{ij}^{(d)} = \frac{1}{2}\left(2(\sigma_{xz}^{(d)})^2\right) = (\sigma_{xz}^{(d)})^2. \tag{5.24}$$

With Equation (5.23a) we obtain

$$\tau = \left|\sigma_{xz}^{(d)}\right| = \rho g(H - z)\sin\alpha. \tag{5.25}$$

After having determined the deviatoric stresses and the effective stress, we can calculate the strain rates. The only non-zero term of the strain rate tensor is

$$\begin{aligned} \dot{\varepsilon}_{xz} &= A\tau^{n-1}\sigma_{xz}^{(d)} = A\left(\rho g(H - z)\sin\alpha\right)^{n-1}\rho g(H - z)\sin\alpha \\ &= A\left(\rho g(H - z)\sin\alpha\right)^{n}. \end{aligned} \tag{5.26}$$

The velocities can be calculated from the strain rates

$$\dot{\varepsilon}_{xz} = \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

where the second term vanishes. Integration with respect to z leads to

$$\begin{aligned} v_x(z) &= 2\int_0^z \dot{\varepsilon}_{xz}(z)\ dz \\ &= -\frac{2A}{n+1}(\rho g \sin\alpha)^n (H-z)^{n+1} + k \end{aligned}$$

With the boundary condition at the glacier base $v_x(0) = u_b$ we can determine the constant k as

$$k = \frac{2A}{n+1}(\rho g \sin\alpha)^n H^{n+1} + u_b$$

and finally arrive at the velocity distribution in a parallel sided slab

$$u(z) = v_x(z) = \underbrace{\frac{2A}{n+1}(\rho g \sin\alpha)^n \left(H^{n+1} - (H-z)^{n+1}\right)}_{\text{deformation velocity}} \quad + \quad \underbrace{u_b}_{\text{sliding velocity}} \tag{5.27}$$

This is Equation (**??**) that we have used before. It is also known as *shallow ice equations*, since it can be shown by rigorous scaling arguments that the longitudinal stress gradients $\frac{\partial \sigma_{xi}}{\partial x_i}$ and $\frac{\partial \sigma_{yi}}{\partial x_i}$ are negligible compared to the shear stress for shallow ice geometries such as the inland parts of ice sheets (except for the domes).