Appendix C

Vectors and Tensors

C.1 Vectors

A vector \( \mathbf{x} \) can be represented in a (not necessarily orthogonal) coordinate system

\[
\mathbf{x} = x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2 + x_3 \hat{\mathbf{e}}_3 \\
= x_p \hat{\mathbf{e}}_p 
\]  

(C.1)

where \( \hat{\mathbf{e}}_i \) are unit length basis vectors. They form a base for a cartesian (orthogonal) coordinate system if all of the following conditions are fulfilled

\[
\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}. 
\]  

(C.2)

The symbol \( \delta_{ij} \) is the Kronecker symbol and is defined by

\[
\delta_{ij} := \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}
\]

In Equation (C.1) we have used the summation convention: we sum over all indices that appear twice.

We now consider another orthogonal coordinate system \( K' \) that is rotated with respect to the original coordinate system \( K \), but has the same origin. The new base vectors \( \hat{\mathbf{e}}'_i \) also fulfill the condition

\[
\hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}'_j = \delta_{ij}. 
\]  

(C.3)

The vector \( \mathbf{x} \) can then be written in the new base as

\[
\mathbf{x} = x'_1 \hat{\mathbf{e}}'_1 + x'_2 \hat{\mathbf{e}}'_2 + x'_3 \hat{\mathbf{e}}'_3 \\
= x'_p \hat{\mathbf{e}}'_p. 
\]  

(C.4)

The vector \( \mathbf{x} \) stays the same, its representation is different in both coordinate systems (the components \( x_i \) and \( x'_i \) are different).
Rotation matrix

We now derive the connection between the representations of the vector $\mathbf{x}$ in both coordinate systems. For this we first define the direction cosine

$$\alpha_{ij} := \hat{e}_i' \cdot \hat{e}_j = \cos(\hat{e}_i', \hat{e}_j)$$  \hspace{1cm} \text{(C.5)}

The quantity $\alpha_{ij}$ is the scalar product of the unit vectors $\hat{e}_i'$ and $\hat{e}_j$ and therefore also the cosine of the angle between the vectors $\hat{e}_i'$ and $\hat{e}_j$ (remember $\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$).

With help of the summation convention (Eq. C.1) we write

$$\mathbf{x} = x_p \hat{e}_p,$$  \hspace{1cm} \text{(C.6)}

an make the scalar product with the unit vector $\hat{e}_i$

$$\mathbf{x} \cdot \hat{e}_i = x_p \hat{e}_p \cdot \hat{e}_i = x_p \delta_{pi} = x_i$$  \hspace{1cm} \text{(C.7)}

and therefore

$$x_i = \mathbf{x} \cdot \hat{e}_i = x_p' \hat{e}_p' \cdot \hat{e}_i = x_p' \alpha_{pi} \cdot$$  \hspace{1cm} \text{(Equation C.4)}  \hspace{1cm} \text{(Equation C.5)}  \hspace{1cm} \text{(C.8)}

Therefore we have shown that the representations of the vector $\mathbf{x}$ in both coordinate systems $K$ and $K'$ are linked by

$$x_i = \alpha_{pi} x_p'.$$  \hspace{1cm} \text{(C.9)}
It can also be shown that the inverse transformation is given by
\[ x'_i = \alpha_{ip}x_p. \] (C.10)

We now derive the same rules for the direction cosines \( \alpha_{ij} \). First we write
\[ x'_i = x_p \alpha_{ip} \quad \text{(Equation C.10)} \]
\[ = \alpha_{jp} x'_j \alpha_{ip} \quad \text{(Equation C.9)} \]
and use \( x'_i = \delta_{ij} x'_j \) so that we obtain \( \delta_{ij} x'_j = \alpha_{jp} \alpha_{ip} x'_j \). This is true for all values of \( x'_j \) so that we arrive at
\[ \alpha_{ip} \alpha_{jp} = \delta_{ij}. \] (C.11)
Similarly it can be shown that
\[ \alpha_{pi} \alpha_{pj} = \delta_{ij}. \] (C.12)

It is convenient to write the direction cosines as a matrix
\[ [\alpha_{ij}] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \] (C.13)

Equations (C.11) and (C.12) can then be written more compactly as
\[ \alpha_{ip} \alpha_{jp} = \delta_{ij} \quad \text{or} \quad [\alpha_{ij}] [\alpha_{ij}]^T = 1, \]
\[ \alpha_{pi} \alpha_{pj} = \delta_{ij} \quad \text{or} \quad [\alpha_{ij}]^T [\alpha_{ij}] = 1, \] (C.14)
and Equation (C.9) is written as
\[ x = [\alpha_{ij}]^T x'. \] (C.15)

The matrix \([\alpha_{ij}]\) is called the rotation matrix. As Equation (C.14) shows, it has the important property that the transpose of the rotation matrix is identical to its inverse
\[ [\alpha_{ij}]^T = [\alpha_{ij}]^{-1}. \] (C.16)

The inverse matrix \([\alpha_{ij}]^{-1}\) is defined through
\[ [\alpha_{ij}] [\alpha_{ij}]^{-1} = [\alpha_{ij}]^{-1} [\alpha_{ij}] = 1. \] (C.17)

**Example** We assume that the coordinate system \( K' \) is rotated by the angle \( \theta \) with respect to the coordinate system \( K \) (Figure C.1). The components of the rotation matrix can be obtained by calculating the scalar products \( \hat{e}_i \cdot \hat{e}'_j \):

<table>
<thead>
<tr>
<th>( \hat{e}'_1 )</th>
<th>( \hat{e}'_2 )</th>
<th>( \hat{e}'_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{e}_1 )</td>
<td>( \cos \theta )</td>
<td>( \sin \theta )</td>
</tr>
<tr>
<td>( \hat{e}_2 )</td>
<td>( -\sin \theta )</td>
<td>( \cos \theta )</td>
</tr>
<tr>
<td>( \hat{e}_3 )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
For the calculation of the components $\alpha_{21}$ and $\alpha_{12}$ in the table we have made use of the relations

\begin{align*}
\cos(\theta - \pi/2) &= \sin(\theta) \\
\cos(\theta + \pi/2) &= -\sin(\theta)
\end{align*}

For example

\[\alpha_{21} = \cos(\hat{e}_{2}', \hat{e}_{1}) = \cos(-\pi/2 - \theta) = -\sin(\theta).\]

Therefore the rotation matrix is

\[
[\alpha_{ij}] = \begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix} \tag{C.18}
\]

Next we take a point $P$ which position vector has the coordinates $(2,1,3)$ in the coordinate system $K$

\[P = 2\hat{e}_1 + \hat{e}_2 + 3\hat{e}_3.\]

We want to calculate the coordinates of $P$ in the system $K'$. With the relation (Eq. C.10)

\[x'_i = \alpha_{ip}x_p\]

we obtain

\begin{align*}
x'_1 &= \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 \\
&= \cos \theta \cdot 2 + \sin \theta \cdot 1 + 0 \cdot 3 \\
x'_2 &= \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 \\
&= -\sin \theta \cdot 2 + \cos \theta \cdot 1 + 0 \cdot 3 \\
x'_3 &= \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3 \\
&= 0 \cdot 2 + 0 \cdot 1 + 1 \cdot 3,
\end{align*}

and arrive at

\[P = (2\cos \theta + \sin \theta)\hat{e}_1' + (-2\sin \theta + \cos \theta)\hat{e}_2' + 3\hat{e}_3'.\]
Comma notation

We consider the scalar field \( f = f(x) \) (e.g. temperature field), the vector field \( f_k = f_k(x) \) (e.g. velocity field) and the tensor field \( f_{pq} = f_{pq}(x) \) (e.g. stress field). We introduce the compact **comma notation** \( f_i \) for the derivative of \( f \) with respect to spatial direction \( x_i \)

\[
f_i := \frac{\partial f}{\partial x_i} \quad i = 1, 2, 3.
\]

Examples of the comma notation

(I) \( (x_k f_k)_i = f_i + x_k f_{k,i} \)

(II) \( (x_k f_k)_{ij} = f_{i,j} + f_{j,i} + x_k f_{k,ij} \)

Proofs:

(I) \( (x_k f_k)_i = x_{k,i} f_k + x_k f_{k,i} \)
\[= \delta_{ki} f_k + x_k f_{k,i} \]
\[= f_i + x_k f_{k,i} \]

(II) \( (x_k f_k)_{ij} = (f_i + x_k f_{k,i})_{,j} \)
\[= f_{i,j} + x_{k,j} f_{k,i} + x_k f_{k,ij} \]
\[= f_{i,j} + f_{j,i} + x_k f_{k,ij} \]
C.2 Tensors

First order tensor

A vector $a$ can be written in two different bases $\mathbf{e}_k$ and $\mathbf{e'}_k$ as

$$a = a_p \mathbf{e}_p \quad \text{and} \quad a = a'_p \mathbf{e'}_p.$$  \hfill (C.19)

Further valid expressions are

$$a'_i = \alpha_{ip} a_p \quad \text{and} \quad a_i = \alpha_{pi} a'_p,$$  \hfill (C.20)

where $\alpha_{ij}$ are the components of a rotation matrix. We now define a (cartesian) tensor of order 1 as a quantity, which is represented by three real numbers that under the change from the $\{x_i\}$-system to the $\{x'_i\}$-system are transformed as

$$a'_i = \alpha_{ip} a_p.$$  \hfill (C.21)

Second order tensor

The above definition is extendable. We consider two vectors $a$ and $b$. Analogous to Equation (C.20) we can write

$$b'_i = \alpha_{ip} b_p \quad \text{and} \quad b_i = \alpha_{pi} b'_p.$$  \hfill (C.22)

We now form the product $a_i b_j$ and look at its transformation behavior

$$a'_i b'_j = (\alpha_{ip} a_p)(\alpha_{jq} b_q)$$
$$= \alpha_{ip}\alpha_{jq} a_p b_q \quad \text{and}$$
$$a_i b_j = (\alpha_{pi} a'_p)(\alpha_{qj} b'_q)$$
$$= \alpha_{pi}\alpha_{qj} a'_p b'_q.$$  \hfill (C.23)

These equations yield the relation between $a_i b_j$ and $a'_i b'_j$. We now write the product $a_i b_j$ as a new quantity

$$c_{ij} := a_i b_j$$

and also

$$c'_{ij} := a'_i b'_j.$$  \hfill (C.24)

The new quantity $[a_ib_j] = [c_{ij}]$ can be written as $3 \times 3$ matrix, the tensor product

\[\begin{array}{ccc}c_{ij} & = & a_i b_j \\
\end{array}\]
of the vectors \( \mathbf{a} \) and \( \mathbf{b} \)

\[
[a, b] = \begin{pmatrix}
a_1 b_1 & a_1 b_2 & a_1 b_3 \\
a_2 b_1 & a_2 b_2 & a_2 b_3 \\
a_3 b_1 & a_3 b_2 & a_3 b_3
\end{pmatrix} = \mathbf{a} \otimes \mathbf{b}
\]

(tensor product)

With the above definitions, Equations (C.23) can be written

\[
c_{ij} = a_i b_j = \alpha_p \alpha_q a'_p b'_q = \alpha_p \alpha_q c'_{pq} \quad \text{and} \quad c'_{ij} = a'_i b'_j = \alpha_{ip} \alpha_{jq} a_p b_q = \alpha_{ip} \alpha_{jq} c_{pq}.
\]

(C.24)

A tensor of order 2 is defined by this transformation rule, i.e. a tensor \( \mathbf{A} \) of order 2 with the components \([A]_{ij} = A_{ij}\) always transforms like

\[
A'_{ij} = \alpha_{ip} \alpha_{jq} A_{pq}.
\]

(C.25)

Tensor of order \( n \)

**Definition:** Given the \( 3^n \) numbers \( a'_{i_1, i_2, \ldots, i_n} \) that transform as

\[
a'_{i_1, i_2, \ldots, i_n} = \alpha_{i_1, j_1} \alpha_{i_2, j_2} \cdots \alpha_{i_n, j_n} a_{j_1, j_2, \ldots, j_n}
\]

under the change from the cartesian coordinate system \( x_i \) to \( x'_i \). These numbers are called cartesian tensor of order \( n \)

A tensor is defined by its transformation properties. To test whether a given quantity is a tensor, the components have to transform according to Equation (C.26).

**Example:** We take some scalar field \( \phi = \phi(\mathbf{x}) \), which could be for example the temperature field. We define the quantity \( a_i := \phi, i \) (the temperature gradient) in any coordinate system \( K \). Are the three numbers \( (a_1, a_2, a_3) \) the components of a tensor?

To answer this question we inquire the transformation properties of \( a_i \). Following the definition of \( a_i \), which is valid for all coordinate systems, we have

\[
a'_i = \frac{\partial \phi}{\partial x'_i},
\]

and with the chain rule

\[
a'_i = \frac{\partial \phi}{\partial x'_i} = \frac{\partial \phi}{\partial x_k} \frac{\partial x_k}{\partial x'_i} = \phi_k \frac{\partial x_k}{\partial x'_i}.
\]
Equation shows that

\[
\frac{\partial x_k}{\partial x'_i} = \alpha_{ik},
\]

such that

\[
a'_i = \alpha_{ik} \phi_k = \alpha_{ik} a_k.
\]

Comparison with Equation (C.26) shows that \(a_i\) is indeed a tensor of first order.

**Example:** We now figure out how the second order tensor \(\mathbf{A}\) looks in the \(K'\)-system, when it has the following form in the \(K\)-system

\[
\mathbf{A} = \begin{pmatrix}
0 & \gamma & 0 \\
\gamma & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

The \(K'\)-system is rotated with respect to the \(K\)-system about the \(\hat{e}_3\) axis by an angle of \(\theta = \pi/4\).

Since \(\mathbf{A}\) is a second order tensor, we have

\[
a'_{ij} = \alpha_{ik} \alpha_{jl} a_{kl} \quad \text{with} \quad [\alpha_{ij}] = \begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

with \(\theta = \pi/4\). Using the transformation formula we obtain

\[
a'_{11} = \alpha_{1k} \alpha_{1l} a_{kl} = \alpha_{11}(\alpha_{11} a_{11} + \alpha_{12} a_{12} + \alpha_{13} a_{13}) + \alpha_{12}(\alpha_{11} a_{21} + \alpha_{12} a_{22} + \alpha_{13} a_{23}) + \alpha_{13}(\alpha_{11} a_{31} + \alpha_{12} a_{32} + \alpha_{13} a_{33}) = \cos \theta(0 \cos \theta + \gamma \sin \theta + 0) + \sin \theta(\gamma \cos \theta + 0 \sin \theta + 0) + 0 (\ldots)
\]

\[
= 2\gamma \cos \theta \sin \theta = 2\gamma \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \gamma.
\]

Therefore we have

\[
a'_{11} = \gamma.
\]
Similarly for the next component we get
\[ a'_{12} = \alpha_{1k} \alpha_{2l} a_{kl} \]
\[ = \alpha_{11} (\alpha_{21} a_{11} + \alpha_{22} a_{12} + \alpha_{23} a_{13}) \]
\[ + \alpha_{12} (\alpha_{21} a_{21} + \alpha_{22} a_{22} + \alpha_{23} a_{23}) \]
\[ + \alpha_{13} (\alpha_{21} a_{31} + \alpha_{22} a_{32} + \alpha_{23} a_{33}) \]
\[ = \cos \theta (-0 \sin \theta + \gamma \cos \theta + 0) \]
\[ + \sin \theta (-\gamma \sin \theta + 0 \cos \theta + 0) \]
\[ + 0 (\ldots) \]
\[ = \gamma (\cos^2 \theta - \sin^2 \theta) \]
\[ = \gamma (\cos^2 \theta - \sin^2 \theta) \big|_{\theta = \pi/4} = 0, \]
and therefore
\[ a'_{12} = \gamma. \]

With this we arrive at the representation of tensor \(A\) in the \(K'\) system
\[
A = \begin{pmatrix}
\gamma & 0 & 0 \\
0 & -\gamma & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Invariants

A quantity which is independent of the orientation of the coordinate system is called an **invariant**. This is best explained with some examples.

**Example:** $a$ and $b$ are vectors with components $a_i$ and $b_j$. Show that the scalar product $c = a \cdot b$ is invariant.

To show this we have to prove that the scalar product is independent of the orientation of the coordinate system, i.e. that $c' = a'_i b'_i$ and $c = a_i b_i$ are the same

$$c' = a'_i b'_i$$
$$= \alpha_{ip} a_p \alpha_{iq} b_q$$
$$= \alpha_{ip} \alpha_{iq} a_p b_q$$
$$= \delta_{pq} a_p b_q$$
$$= a_p b_p$$
$$= a_i b_i$$
$$= c$$

**Example:** $c_{ij}$ are the components of a second order tensor. Show that the trace $c_{ii}$ is an invariant.

$$c'_{ii} = \alpha_{ip} \alpha_{iq} c_{pq}$$
$$= \delta_{pq} c_{pq}$$
$$= c_{pp} = c_{ii}$$

**Example:** Show that $c_{ik} c_{ki}$ is an invariant.

$$c'_{ik} c'_{ki} = \alpha_{iq} \alpha_{kp} c_{qp} \alpha_{kr} \alpha_{is} c_{rs}$$
$$= \alpha_{iq} \alpha_{is} \alpha_{kp} \alpha_{kr} c_{qp} c_{rs}$$
$$= \delta_{qs} \delta_{pr} c_{qp} c_{rs}$$
$$= c_{qp} c_{pq} = c_{ik} c_{ki}$$
Appendix C Vectors and tensors

The permutation symbol $\epsilon_{ijk}$

The **permutation symbol** $\epsilon_{ijk}$ has the value zero if at least two of the indices are the same. Otherwise the symbol has the value +1 or -1, depending on whether the order of the indices is cyclic or anticyclic. Thus we have $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ (cyclic indices), $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$ (anticyclic indices), and $\epsilon_{iij} = \epsilon_{ijj} = \epsilon_{jii} = 0$ (no summation over repeated indices!) for $i$ and $j = 1, 2, 3$.

$$\epsilon_{ijk} := \begin{cases} +1, & i, j, k \text{ in cyclic order} \\ -1, & i, j, k \text{ in anticyclic order} \\ 0, & \text{two or more indices have the same value} \end{cases}$$

The permutation symbol is also known as the Levi-Civita $\varepsilon$ symbol. This symbol is mainly used to write the vector product in index notation. One valid expression is

$$\epsilon_{ijk} = \hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k).$$

A useful relation between the Kronecker symbol and the permutation symbol is the $\delta - \varepsilon$ relation

$$\epsilon_{ijk} \delta_{kpq} = \delta_{ip} \delta_{jq} - \delta_{jp} \delta_{iq}. \quad (C.27)$$

Isotropic tensors

An **isotropic tensor** is a tensor with the same entries in each coordinate system. For each isotropic tensor of order $n$ this relation holds

$$A'_{ijk...} = A_{ijk...}.$$

The unit tensor $\delta_{ij}$ is an example of an isotropic tensor, since

$$\delta'_{ij} = \alpha_{ip} \alpha_{jq} \delta_{pq} \quad (\text{Eq. C.25})$$

$$= \alpha_{ip} \alpha_{jp} \quad (\text{Eq. C.11}),$$

and therefore $\delta'_{ij} = \delta_{ij}$. 

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We note some relations without proof, that will be useful in further sections.

- Each isotropic second order tensor $\mathbf{A}$ with components $[\mathbf{A}]_{ij}$ can be written in the form
  
  $$[\mathbf{A}]_{ij} = \alpha \delta_{ij}$$
  
  where $\alpha$ is a scalar.

- Each isotropic third order tensor $\mathbf{A}$ with components $[\mathbf{A}]_{ijk}$ can be written in the form
  
  $$[\mathbf{A}]_{ijk} = \alpha \varepsilon_{ijk}$$
  
  where $\alpha$ is a scalar.

- Each isotropic fourth order tensor $\mathbf{A}$ with components $[\mathbf{A}]_{ijkl}$ can be written in the form
  
  $$[\mathbf{A}]_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$$
  
  where $\alpha$, $\beta$ and $\gamma$ are scalar quantities.