Instability in glacial systems

Martin Lüthi

Versuchsanstalt für Wasserbau, Hydrologie und Glaziologie (VAW)
ETH Zürich, 8092 Zürich, Switzerland

Introduction

In his illustrative and insightful 1981 paper “Eislawinen und Ausbrüche von Gletscherseen”, Hans Röthlisberger reports on the most dramatic changes in glaciers: ice avalanches and outburst floods of glacier lakes. Both phenomena are rare and sudden changes in the dynamics and geometry of the glacier that occur during very short time intervals compared with the time scales of their posterior evolution. The long term evolution of these glaciers is controlled in large part by the rare catastrophic events.

The investigation and prediction of glacier hazards has been one of the key activities of the VAW Glaciology Section for the last 40 years. Ever since the days of Hans Röthlisberger, hazard mitigation studies for power companies, touristic railway companies and governmental agencies have been part of the daily business at VAW. Many important results of these studies have been published (e.g. Haefeli, 1965; Röthlisberger, 1977; Iken, 1977; Spring and Hutter, 1981; Röthlisberger, 1987; Röthlisberger, 1993; Lüthi and Funk, 1997; Wegmann and others, 1998; Funk and Margreth, 1999; Margreth and Funk, 1999; Pralong and others 2003), an even larger fraction is documented in internal reports.

Ice avalanches

Confronted with the problem to predict the time of release of a pending large ice mass from a hanging glacier at Weisshorn (Wallis, Switzerland), Flotron (1977)
and Röthlisberger (1977) proposed the now famous empirical relation for the ice velocity before breakoff

\[ v(t) = v_o + a(t_f - t)^{-m}, \quad m > 0. \]  

(1)

The velocity \( v(t) \) increases from its initial value \( v_o \) to infinity at the time of failure \( t_f \). The velocity measurements from Weisshorn in Figure 1 clearly illustrate the progressive increase of velocity prior to break-off (these were the first detailed velocity measurements of an instable hanging glacier).

**Finite-time singularities**

Finite-time singularities arise in systems with strong feedback mechanisms (e.g. Voight, 1989). The occurrence of a finite-time singularity is the signature of a transition to
another regime – here the transition from (steady) viscous flow to (unstable) damage or fracture processes within the glacier. The simplest system that captures the essence of the finite-time singularity consists of some quantity \( p \) with a positive feedback on its growth rate

\[
\frac{dp}{dt} = p^k, \quad k > 1, \quad \text{with the solution} \quad p(t) = p(0) \left( \frac{t_f - t}{t_f} \right)^{-\frac{1}{k-1}}. \tag{2}
\]

We note that Equation (2) is self-similar or "scale invariant" which means that it keeps the same form under rescaling of the time. Under the transformation \((t_f - t) \to \lambda (t_f - t)\) the function \( p(t) \) is rescaled to \( \lambda^{-(1/(k-1))} p(t) \).

**Conceptual model of a hanging glacier**

We investigate the mechanism leading to the acceleration prior to failure of a hanging glacier with a simplified model. More realistic and sophisticated numeric models have been developed, based on progressive fracturing (Iken, 1977) and damage mechanics (Pralong and others, 2003). Here we describe the mechanism of damage accumulation in a local formulation without considering the geometry of the unstable glacier parts.

Despite its simplicity, the model exhibits the most important feature of the growth of a crevasse of depth \( h \) in a glacier of thickness \( H \), as sketched in Figure 2a. The overhanging serac induces a locally unbalanced stress \( \tau_o \) at the glacier front which is the driving force of crevasse formation and growth. The relative depth of the crevasse \( \omega = h/H \) is a measure of material deterioration, or damage. Further we assume that the ice velocity \( v \) and crevasse growth are depended on the stress state \( \tau \) in the vicinity of the crevasse bottom (the “working zone” where fracturing or damage occur). Further we assume that the geometry is unchanged during the failure process, and the initial load \( \tau_o \) is constant. This leads to the following equation system:

\[
\text{stress in damaged ice} \quad \tau = \tau_o \frac{H}{H - h} = \frac{\tau_o}{1 - \omega}, \tag{3}
\]

\[
\text{damage evolution} \quad \frac{d\omega}{dt} = D \frac{\tau_o^r}{(1 - \omega)^k}, \tag{4}
\]

\[
\text{ice deformation} \quad v = F \tau^n. \tag{5}
\]

The equation system (3) – (5) describes local damage evolution with one damage state variable \( \omega \) (Kachanov-Rabotnov theory, e.g. Skrzypek and Ganczarski,
Figure 2: a) Sketch of the crevasse formation due to a pending crevasse. $H$ is total ice thickness, $h$ the depth of the crevasse, $\tau_o$ the locally unbalanced stress due to the overhanging serac and $\tau$ the stress in the “working zone” (shaded). b) Sketch of the “working zone” (shaded) at the glacier base where damage accumulates.

1999). The first relation states that the stress $\tau_o$ is supported by the fraction of undamaged ice in the “working zone”. As the crevasse deepens (the damaging proceeds), the area supporting $\tau_o$ is reduced and therefore $\tau$ increases. Equation (3) is the definition of the damage $\omega$ we employ. The second relation is a subcritical damage (or crack) evolution law and the third relation describes ice velocity due to deformation in the “working zone”. The parameter $F$ includes the rate factor of ice deformation and a “form factor” describing the geometry of the “working zone”.

Equations (3) and (4) contain the feedback that leads to the finite-time singularity

$$\omega(t) = 1 - [(k + 1)D\tau_o^r]^{\frac{1}{k+1}} (t_f - t)^{\frac{1}{k+1}}.$$  
(6)

Under the assumption that $t_o = 0$, $\omega(t_o) = 0$, and that ultimate failure occurs at time $t_f$ when the damage $\omega(t_f) = 1$, the duration of the acceleration phase prior to failure is

$$t_f = [(k + 1)D\tau_o^r]^{-1}.$$  
(7)

The failure time scale $t_f$ depends only on $\tau_o$ and the parameters of damage evolution (Eq. 4). The evolution of damage and stress can be expressed as

$$\omega(t) = 1 - \left(\frac{t_f - t}{t_f}\right)^{\frac{1}{k+1}} \quad \text{and} \quad \tau(t) = \tau_o \left(\frac{t_f - t}{t_f}\right)^{-\frac{1}{k+1}},$$  
(8)

and inserted into (5) yield the velocity

$$v(t) = F\tau_o^m \left(\frac{t_f - t}{t_f}\right)^{-m} = a(t_f - t)^{-m},$$  
(9)
where \( m = n/(k + 1) \) and \( a = F \tau_o^n t_f^m \). Equation (9) has the same form as the empirical law (1) except for the initial velocity \( v_o \) which describes the glacier movement prior to initiation of the damage process.

The simple model provides a physical interpretation of the empirical parameters \( a \) and \( m \) of Equation (1). While \( m \) depends on the exponents of the flow and damage laws, \( a \) depends on both rate factors \( F \) and \( D \), the initial stress \( \tau_o \) and the exponents.

### Characteristic scales

The parameter \( a \) in Equation (9) contains characteristic length and time scales of the failure process

\[
a = F \tau_o^n t_f^m = v_{ss} t_f^m, \tag{10}
\]

where \( v_{ss} = F \tau_o^n \) is the steady state velocity without damage processes (which should be equal to \( v_o \)). Time-integration of Equation (9) gives the position of the accelerating instable ice mass (assuming \( s(0) = 0 \))

\[
s(t) = s_f - b (t_f - t)^q, \tag{11}
\]

where \( q = 1 - m \), \( b = a/q \) and \( s_f \) is the position when failure occurs

\[
s_f = b (t_f)^q = \frac{F \tau_o^n t_f}{q} = \frac{v_{ss} t_f}{q}. \tag{12}
\]

Parameter \( b \) (and therefore \( a \)) can be expressed in terms of the characteristic scales

\[
b = s_f \left( \frac{1}{t_f} \right)^q. \tag{13}
\]

Equation (13) predicts that \( b \) depends on \( q \). Such a dependence has been detected in data from instable hanging glaciers. Survey data from 14 break-off events have been fitted with Equation (11) (unpublished data from H. Röthlisberger, A. Flotron, M. Funk, A. Pralong). Each fit is represented with a dot in the parameter space in Figure 3. All except two points are on the a curve (solid line in Fig. 3) which can be approximated with

\[
b(q) \simeq 55 \left( \frac{1}{550} \right)^q. \tag{14}
\]

The data quality of both break-off events E* and J* that are below the approximating curve in Figure 3 is unsatisfactory. Only four positions have been measured
Figure 3: Dependence of the parameter $b$ on $q$. The dots indicate parameter pairs of fitted time-to-failure curves of several hanging glaciers. Letters indicate the glaciers (M: Mönch, E: Eiger, J: Jorasses, W: Weisshorn, G: Gutz) and stars denote small lamellas. The solid line is the best fit through the data points. The data point at $(b, q) = (35, 0.09)$ lays on the curve as well.

at the small lamella $J^*$, resulting in large uncertainties. The measurements of $E^*$ stopped 30 days before breakoff. It is possible that the geometry changed, or small parts broke off prior to the main failure.

Identification of corresponding terms in Equations (13) and (14) yields

$$ t_f = 550 \text{ d}, \quad s_f = 55 \text{ m} \quad \text{and} \quad \frac{v_{ss}}{q} = \frac{s_f}{t_f} = 0.1 \text{ m d}^{-1}. \quad (15) $$

The characteristic time scale $t_f$ seems to be universal and independent of the individual glacier geometry. The universality may be an indication that the damage process is the same in all observed cases (which is the essence of Eq. 7). Furthermore we may speculate that damaging only occurs above a threshold stress $\tau_{th}$. If
the glacier geometry is constant during the failure process, $\tau_o = \tau_{th}$ is maintained in the “working zone” until failure. This would explain why time to failure is the same in all cases.

The relation $q = 10 v_{ss}$ (Eq. 15) is most remarkable. If the above reasoning is correct, it is sufficient to calculate the steady ice velocity due to deformation in the “working zone” at the threshold stress to obtain an estimate of $q$ which in turn allows to calculate $b$. The parameter $q$ is a measure of the “activity” of a hanging glacier. Indeed, Equation (12) can be rewritten as

$$q = \frac{v_{ss} t_f}{s_f},$$

and relates the position that would have been attained without damage processes within the time span $t_f$ to the position $s_f$ attained just before break-off with damaging.

We shortly resume what we have accomplished so far. Through theoretical reasoning and empirical data analysis we found that two fitting parameters $b$ and $q$ are correlated. This considerably reduces the complexity of predicting the failure time $t_f$. Furthermore we have found that $t_f = 550 \text{ d}$ and $s_f = 55 \text{ m}$ are universal constants of the damage process in hanging glaciers. The physical meaning of relation (15) is still somewhat mysterious and deserves further investigation.

### Log-periodic corrections

The velocity data from the Weisshorn hanging glacier (Figure 1) show an oscillating behavior modulated on the singular increase prior to break-off. Similar oscillations are visible in survey data from instable parts of other hanging glaciers. Log-periodic oscillations that decorate the leading singular evolution have been detected in many natural and engineering systems prior to ultimate failure (e.g. Sornette and Sammis, 1995; Johansen and Sornette, 1998). The oscillation frequency decreases logarithmically as the time of failure is reached (thus the name “log-periodic oscillations”)

$$s(t) = s_f - b(t_f - t)^q \left(1 + C \cos \left(2\pi \frac{\ln(t_f - t)}{\ln \lambda} + D\right)\right).$$

The best log-periodic fit to the data from Weisshorn is shown with a dotted line in Figure 1. The agreement with the measured velocity is reasonable with a time scale parameter $\lambda \sim 1.7$. 

---

*Log-periodic corrections*

The velocity data from the Weisshorn hanging glacier (Figure 1) show an oscillating behavior modulated on the singular increase prior to break-off. Similar oscillations are visible in survey data from instable parts of other hanging glaciers. Log-periodic oscillations that decorate the leading singular evolution have been detected in many natural and engineering systems prior to ultimate failure (e.g. Sornette and Sammis, 1995; Johansen and Sornette, 1998). The oscillation frequency decreases logarithmically as the time of failure is reached (thus the name “log-periodic oscillations”)

$$s(t) = s_f - b(t_f - t)^q \left(1 + C \cos \left(2\pi \frac{\ln(t_f - t)}{\ln \lambda} + D\right)\right).$$

The best log-periodic fit to the data from Weisshorn is shown with a dotted line in Figure 1. The agreement with the measured velocity is reasonable with a time scale parameter $\lambda \sim 1.7$. 

---

*Log-periodic corrections*

The velocity data from the Weisshorn hanging glacier (Figure 1) show an oscillating behavior modulated on the singular increase prior to break-off. Similar oscillations are visible in survey data from instable parts of other hanging glaciers. Log-periodic oscillations that decorate the leading singular evolution have been detected in many natural and engineering systems prior to ultimate failure (e.g. Sornette and Sammis, 1995; Johansen and Sornette, 1998). The oscillation frequency decreases logarithmically as the time of failure is reached (thus the name “log-periodic oscillations”)

$$s(t) = s_f - b(t_f - t)^q \left(1 + C \cos \left(2\pi \frac{\ln(t_f - t)}{\ln \lambda} + D\right)\right).$$

The best log-periodic fit to the data from Weisshorn is shown with a dotted line in Figure 1. The agreement with the measured velocity is reasonable with a time scale parameter $\lambda \sim 1.7$. 

---

*Log-periodic corrections*

The velocity data from the Weisshorn hanging glacier (Figure 1) show an oscillating behavior modulated on the singular increase prior to break-off. Similar oscillations are visible in survey data from instable parts of other hanging glaciers. Log-periodic oscillations that decorate the leading singular evolution have been detected in many natural and engineering systems prior to ultimate failure (e.g. Sornette and Sammis, 1995; Johansen and Sornette, 1998). The oscillation frequency decreases logarithmically as the time of failure is reached (thus the name “log-periodic oscillations”)

$$s(t) = s_f - b(t_f - t)^q \left(1 + C \cos \left(2\pi \frac{\ln(t_f - t)}{\ln \lambda} + D\right)\right).$$

The best log-periodic fit to the data from Weisshorn is shown with a dotted line in Figure 1. The agreement with the measured velocity is reasonable with a time scale parameter $\lambda \sim 1.7$. 

---

*Log-periodic corrections*

The velocity data from the Weisshorn hanging glacier (Figure 1) show an oscillating behavior modulated on the singular increase prior to break-off. Similar oscillations are visible in survey data from instable parts of other hanging glaciers. Log-periodic oscillations that decorate the leading singular evolution have been detected in many natural and engineering systems prior to ultimate failure (e.g. Sornette and Sammis, 1995; Johansen and Sornette, 1998). The oscillation frequency decreases logarithmically as the time of failure is reached (thus the name “log-periodic oscillations”)
Equation (17) follows from (11) if the critical exponent $q$ is complex (Sornette, 1998). To see how this is achieved we rescale time $(t_f - t) \rightarrow \lambda(t_f - t)$

$$s(t) = s_f - b(t_f - t)^q = s_f - \frac{b}{\mu} (\lambda(t_f - t))^q \quad \Rightarrow \quad \frac{\lambda^q}{\mu} = 1$$  (18)

the complex solution of which is (using $1 = e^{i2\pi k}$, where $k$ is an integer)

$$q = \frac{\ln \mu}{\ln \lambda} + i \frac{2\pi k}{\ln \lambda}$$  (19)

The special case $k = 0$ gives the usual real power-law solution corresponding to fully continuous scale invariance. The more general complex solution corresponds to a discrete scale invariance with preferred time scale $\lambda$ (Gluzman and Sornette, 2001). The log-periodic solution (17) contains additional parameters $C$ and $\lambda$ (and probably a phase $D$). Its practical advantage is the possibility to predict more accurately the time of failure since the oscillations provide a “lock-in” of the solution. Several attempts have been made to relate log-periodic behavior with systems that contain a relaxation mechanism reducing the damage (e.g. Ide and Sornette, 2002), or with dynamical crack interactions at different scales (Sahimi and Arbabi, 1996). However, the physical meaning of $\lambda$ is not fully revealed by these approaches and further investigations are needed.

**Discussion**

Critical phenomena are often associated with finite-time singularities. The above presentation showed how the progressive damaging of ice in a hanging glacier leads to a velocity increase with a singularity at the time of failure. Similar finite-time singularities occur in the theory of outburst floods from glacier dammed lakes which we outline briefly.

The theory of water flow in subglacial channels was developed by Röthlisberger (1972) and Nye (1976). When the water input to a subglacial channel is maintained at high pressure for a long time, the water discharge $Q$ through the channel evolves as (eq. 30 in Nye, 1976)

$$\frac{dQ}{dt} = K_1 Q^{5/4},$$  (20)

where $K_1$ depends on the channel geometry and roughness. The evolution of discharge during a lake drainage event (Jökullhlaup) follows exactly relation (2). Here, the feedback arises through the unstable growth of the pressurized channel.
due to melting at the channel walls. In Equation (20) the channel closure due to ice deformation has been neglected. When channel closure is included (Clarke, 1982) the emerging equation system contains a negative feedback which leads to oscillatory behavior (Ng and Björnsson, 2002).

Conclusions

Instable ice masses on hanging glaciers exhibit a characteristic acceleration before break-off. With help of a simplified model several remarkable properties of the progressive damaging prior to failure have been identified. The empirical formula used to predict failure time was derived, and a physical interpretation was given. The solution revealed that two parameters of the empirical formula are related. This considerably reduces the complexity of predicting the time of failure. Characteristic length and time scales have been found, which seem to be universal quantities of the failure process in hanging glaciers. The numerical values $t_f = 550 \text{ d}$ and $s_f = 55 \text{ m}$ were obtained by analyzing survey data from 14 break-off events.

Measured ice velocities prior to break-off exhibit oscillations which can be described as log-periodic corrections to the leading velocity singularity. The exact cause for these oscillations is still unclear, but dynamical crack interactions or damage reduction due to healing are likely explanation.

The presented simple model of damage evolution results in equations that are similar in structure to those of the theory of instable evolution of subglacial channels, used to describe lake outburst floods. Both phenomena belong to a much wider class of critical processes that are intensely studied. Insight gained in the investigation of one process can reveal new perspectives in the study of other processes, thus leading to a better understanding of instabilities in glaciers.

Acknowledgments

Heartily thanks are due to Hans Röthlisberger for many insightful and amusing discussions. The collaboration with Martin Funk and Antoine Pralong is highly appreciated. Both commented on an earlier version of the manuscript.
References


