On Feature Learning with State Space Models and Pulse Domain Signal Analysis

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based on joint work with
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Two vaguely related pieces of work and a view.
Piece 1 (the solid piece):

**A Single Trick and Algorithm for Many Problems in Signal Analysis**

Combining

- linear state space models,
- normal priors with unknown variances (NUV) for sparsity,
- and expectation maximization (EM) for learning all parameters

can be used for sparse estimation, dictionary learning, unsupervised signal labeling, blind signal separation, and more,

by variations of a single algorithm essentially consisting of repeated multivariate-Gaussian forward-backward message passing (i.e., recursions as in Kalman smoothing).

[ITA 2016], [EUSIPCO 2017], [PhD thesis Zalmai 2017]
Sparsity by NUV Priors (Normal with Unknown Variance)

- Originating from Bayesian inference [MacKay 1992, Neal 1996, ...]
- Basis of “automatic relevance determination” and sparse Bayesian learning [Neal, Tipping 2001, Wipf et al., ...]

Example: real $U \sim \mathcal{N}(0, s^2)$ with unknown variance $s^2$,
single observation $Y = U + Z = \mu \in \mathbb{R}$ with noise $Z \sim \mathcal{N}(0, \sigma^2)$:

Maximum-likelihood estimate $\hat{s}_{ML}^2 = \max\{0, \mu^2 - \sigma^2\}$
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$$
\begin{align*}
\mathcal{N}(0, 1) & \xrightarrow{s} U \\
\mathcal{N}(0, \sigma^2) & \xrightarrow{\mu} \mathcal{N}(\mu, \sigma^2)
\end{align*}
$$

Maximum-likelihood estimate $s^2_{\text{ML}} = \max\{0, \mu^2 - \sigma^2\}$

For fixed $s^2 = s^2_{\text{ML}}$, $U$ is Gaussian with posterior mean (MAP/MMSE/LMMSE estimate)

$$
\hat{u} = \begin{cases} 
\mu \cdot \frac{\mu^2 - \sigma^2}{\mu^2} & \text{if } \mu^2 > \sigma^2 \\
0, & \text{otherwise}
\end{cases}
$$
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    0, & \text{otherwise.}
  \end{cases}
  \]
- Still holds for $Y \in \mathbb{R}^N$ with likelihood $p(y|u) \propto e^{-(u-\mu(y))^2/2\sigma^2}$. 
Sparsity by NUV Priors cont’d

General method:

- Model variables (or parameters) $U_1, \ldots, U_K$ of interest as independent zero-mean Gaussians, each with its own individual unknown variance $\sigma^2_1, \ldots, \sigma^2_K$.

- Determine $\sigma^2_1, \ldots, \sigma^2_K$ by ML (or some approximation thereof); e.g., by expectation maximization (EM). A local maximum of the likelihood suffices for sparsity.

Specifically (for linear Gaussian models):

1. Begin with an initial guess $\hat{\sigma}^2_1, \ldots, \hat{\sigma}^2_K$.

2. Compute* the means $m_{U_k}$ and the variances $\sigma^2_{U_k}$ of the (Gaussian) posterior distributions $p(u_k | y, \sigma^2_1, \ldots, \sigma^2_K)$ for $k = 1, \ldots, K$ with $\sigma^2_1, \ldots, \sigma^2_K$ fixed.

3. Standard EM: update $\sigma^2_k \leftarrow m^2_{U_k} + \sigma^2_{U_k}$ for all $k$.

4. Repeat 2 and 3 until convergence.

*by Gaussian message passing in the appropriate factor graph
Linear State Space Models

State $X_k \in \mathbb{R}^n$ and observation $Y_k \in \mathbb{R}^L$ evolving according to

\[
X_k = AX_{k-1} + BU_k \\
Y_k = CX_k + Z_k
\]

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{L \times n}$, and where $U_k$ (with values in $\mathbb{R}^m$) and $Z_k$ (with values in $\mathbb{R}^L$) are independent zero-mean white Gaussian noise processes.

Factor graph:
Linear state space models with sparse input: Sparse Scalar Input

E.g.:

- Sparse input-signal estimation (e.g., heart beat [ISIT 2015]):

- Piecewise constant least-squares fit:
Linear state space models with sparse input:

**White Noise Input + Sparse Scalar Input**

\[ N \] \[ U_{k,1} \]
\[ B \]
\[ \sigma_k \]
\[ \times \]
\[ U_{k,2} \]
\[ b \]
\[ + \]
\[ X_k \]
\[ + \]
\[ X_k \]

instead of

\[ N \]
\[ U_k \]
\[ B \]
\[ + \]
\[ X_k \]

E.g., random walk with occasional jumps:
Linear state space models with sparse input:

**Multiple Sparse Scalar Inputs**

\[
\begin{aligned}
N^r & \quad \sigma_{k,1} \quad \times \quad U_{k,1} \quad b_1 \\
& + \quad X_k \\
\end{aligned}
\]

\[
\begin{aligned}
N^r & \quad \sigma_{k,2} \quad \times \quad U_{k,2} \quad b_2 \\
& + \quad X_k \\
\end{aligned}
\]

instead of

\[
\begin{aligned}
N^r & \quad U_k \\
& + \quad X_k \\
\end{aligned}
\]

E.g., least-squares fitting of straight-line segments:

Obvious generalizations:

- polynomial segments
- enforcing continuity, or continuity of derivative(s)
Linear state space models with sparse input:

**Dealing with Outliers**

Simply replace $Y = CX_k + Z_k$
by $Y = CX_k + Z_k + \tilde{Z}_k$ with sparse $\tilde{Z}_k$, i.e.,

\[
\cdots \xrightarrow{C} \tilde{Y}_k \xrightarrow{\text{+}} \sigma_k \xrightarrow{\text{+}} \tilde{Z}_k \xrightarrow{\times} y_k \xrightarrow{\text{-}} \cdots
\]

instead of

\[
\cdots \xrightarrow{C} \tilde{Y}_k \xrightarrow{\text{+}} \sigma_k \xrightarrow{\text{+}} Z_k \xrightarrow{\times} y_k \xrightarrow{\text{-}} \cdots
\]

Example:
Linear state space models with sparse input: [ICASSP 2016]

**Sparse Input Pulses with Individual Direction**

\[ \dot{U}_k,1 \times \quad \dot{U}_k,2 \quad B \quad X_k \quad \text{instead of} \quad \dot{U}_k,1 \quad \dot{U}_k,2 \quad B \quad X_k \]

- Unknown scalar \( \sigma_k \) replaced by unknown vector \( b_k \in \mathbb{R}^n \)
- Still sparsifying, still learnable (e.g.) by EM

**Applications:**
- Occasional arbitrary jumps in the state space
- System identification from multiple unknown excitations
- \ldots
Linear state space models with sparse input:

**Recurring Unknown Sparse Input Pulses**

- Unknown input vectors $b_1, \ldots, b_M \in \mathbb{R}^n$, each with independent sparse input.
- Still learnable by EM. The state transition matrix $A$ can also be learned.

Applications: unsupervised signal labeling, dictionary learning, blind signal separation, ...
Unsupervised Feature Extraction, Signal Labeling, and Blind Signal Separation

Artificial example: irregular occurrences of localized signal shapes on top of a wandering baseline with jumps.

Everything (matrices $A$, $B$, input signals) is learned, unsupervised.
Unsupervised Feature Extraction, Signal Labeling, and Blind Signal Separation

ECG recording of a pregnant woman: decomposition into maternal and fetal heart beats.

Totel model order 24: 8 and 3 damped sinusoids, respectively, for the heart beats; local line model (≈ cubic spline) for the baseline.
Multichannel Sparse Impulsive Signals as Data Type for Signal Analysis

The method just described yields sparse multichannel feature “signals”:
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The method just described yields sparse multichannel feature “signals”:

The same method can be applied again to such signals!
(Gaussian estimation \(\approx\) least squares \(\approx\) orthogonal projection.)

(Mentioned in [Zalmai thesis 2017], but no experience as yet.)
Multichannel Sparse Impulsive Signals as Data Type for Signal Analysis

The method just described yields sparse multichannel feature “signals”:

\[ s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6 \quad s_7 \quad s_8 \quad s_9 \quad s_{10} \quad s_{11} \quad s_{12} \quad s_{13} \quad s_{14} \quad s_{15} \quad s_{16} \quad s_{17} \quad s_{18} \quad s_{19} \]

\[ = \]

\[ s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6 \quad s_7 \quad s_8 \quad s_9 \quad s_{10} \quad s_{11} \quad s_{12} \quad s_{13} \quad s_{14} \quad s_{15} \quad s_{16} \quad s_{17} \quad s_{18} \quad s_{19} \]

The same method can be applied again to such signals!

(Gaussian estimation \( \approx \) least squares \( \approx \) orthogonal projection.)

And again, and again . . . , to any depth (all unsupervised).

(Mentioned in [Zalmai thesis 2017], but no experience as yet.)
Layered Networks of Feature Detection Filters

Such multichannel sparse feature signals have already been used in parallel work:

Feature detection filters ("neurons") work here as follows:

- A multi-input, single-output linear time-invariant filter (IIR) produces a score signal (= correlation with a smooth template).
- An isolated unit pulse is generated if the score signal exceeds some threshold. (Sparsity is essential: thresholding does not work.)
Piece 2: Layered Networks of Feature Detection Filters

Toy Example of Three-Channel Template

- Time scale: at most one pulse in window
- Realizable with biological plausible neurons
- Realizable with simple analog circuits
Piece 2:

**Layered Networks of Feature Detection Filters**

Feature detection filters ("neurons"):

- Score signal (= correlation with smooth template) is computed by IIR filter.
- An isolated unit pulse is generated if the score signal exceeds some threshold.
- Allows biologically plausible neuron models.
- Supervised learning of deep network based on gradient back-propagation demonstrated (for toy example), apparently avoiding gradient degeneration [Neff thesis 2016].
- Promising for (non-digital) neuromorphic computation.
Conclusion

The solid piece:

- Linear state space models with NUV priors can be used for sparse estimation, dictionary learning, unsupervised signal labeling, blind signal separation, ...

- ... by variations of a single algorithm consisting essentially of repeated multivariate-Gaussian forward-backward message passing (i.e., recursions as in Kalman smoothing).

The view:

Sparse multichannel feature signals are an interesting data type for signal analysis. Features-of-features networks with such signals can be built as in Piece 1 or as in Piece 2.

(Did not discuss relations to convolutional neural networks, wavelets, ... )
Main References


- Zalmai, Keusch, Malmberg, Loeliger, “Unsupervised feature extraction, signal labeling, and blind signal separation in a state space world,” EUSIPCO 2017


- Loeliger and Neff, “Pulse-domain signal parsing and neural computation,” ISIT 2015