Exercise 11 - Linear Systems Analysis

Up to this point we have gathered different tools for modeling various types of systems and determining their unknown parameters. Also, some insight was provided in the analysis of nonlinear systems. It turns out, however, that working with such systems and trying to understand their behaviour can be very difficult since not many approaches are available. To tackle this problem, they are linearized and, by performing this operation, the analysis becomes easier and some conclusions about the original nonlinear behaviour can be drawn. The nonlinear systems resulting form the modeling take the form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}x(t) &= f(x(t), u(t), t) \\ y(t) &= g(x(t), u(t), t), \end{aligned} \tag{11.1}$$

with

- $x(t) \in \mathbb{R}^n$: state of the system;
- $u(t) \in \mathbb{R}^m$: input to the system;
- $y(t) \in \mathbb{R}^p$: output of the system.

11.1 Normalization

Normalization consists of the scaling of the variables of interest and allows us to avoid numerical issues when simulating and optimizing a given system.
Consider the scaling factors $x_{i,0}, u_{j,0}, y_{k,0}$ and the normalized variables $x_{i,N}(t), u_{j,N}(t), y_{k,N}(t)$ such that

$$\begin{aligned} x_i(t) &= x_{i,0} \cdot x_{i,N}(t) &\rightarrow\quad x_{i,N}(t) &= \frac{x_i(t)}{x_{i,0}} & i &= 1, 2, \dots, n, \\ u_j(t) &= u_{j,0} \cdot u_{j,N}(t) &\rightarrow\quad u_{j,N}(t) &= \frac{u_j(t)}{u_{j,0}} & j &= 1, 2, \dots, m, \\ y_k(t) &= y_{k,0} \cdot y_{k,N}(t) &\rightarrow\quad y_{k,N}(t) &= \frac{y_k(t)}{y_{k,0}} & k &= 1, 2, \dots, p. \end{aligned}$$

The normalization constants are chosen such that the normalized variables have no physical units and have roughly the order of magnitude 1.
We can write this transformation in vector notation as

$$\begin{aligned} x &= T_x \cdot x_N, & T_x &= \operatorname{diag}\{x_{1,0}, \dots, x_{n,0}\}, \\ u &= T_u \cdot u_N, & T_u &= \operatorname{diag}\{u_{1,0}, \dots, u_{m,0}\}, \\ y &= T_y \cdot y_N, & T_y &= \operatorname{diag}\{y_{1,0}, \dots, y_{p,0}\}. \end{aligned}$$

From which we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}x_N &= T_x^{-1} f(T_x \cdot x_N, T_u \cdot u_N, t) =: f_N(x_N, u_N, t) \\ y_N &= T_y^{-1} g(T_x \cdot x_N, T_u \cdot u_N, t) =: g_N(x_N, u_N, t). \end{aligned}$$

Remark. It can be shown that such a transformation does not change the properties of the system.

This material is based on the HS17 teaching assistance taught by Nicolas Lanzetti and Gioele Zardini.
This document can be downloaded at `https://n.ethz.ch/~lnicolas`

11.2 Linearization

Linear dynamical systems can be studied very effectively using tools from linear system theory. Even though linearization is an imperfect representation of general nonlinear systems it can be useful to characterize local behaviour.
Consider a trajectory $\bar{x}(t)$ such that $\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t))$ with given $\bar{u}(t)$. Let

$$\begin{aligned} \delta x(t) &:= x(t) - \bar{x}(t), \\ \delta u(t) &:= u(t) - \bar{u}(t). \end{aligned} \tag{11.2}$$

be *small* perturbations from the trajectory $\bar{x}(t)$ and the input $\bar{u}(t)$, respectively. The perturbation dynamics reads

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\delta x(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) - \bar{x}(t)\right) \\ &= f(x(t), u(t)) - f(\bar{x}(t), \bar{u}(t)) \\ &= f(\bar{x}(t) + \delta x(t), \bar{u}(t) + \delta u(t)) - f(\bar{x}(t), \bar{u}(t)) \\ &\approx \underbrace{\left.\frac{\partial f}{\partial x}\right|_{(\bar{x}(t), \bar{u}(t))}}_{A(t) \in \mathbb{R}^{n \times n}} \delta x(t) + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{(\bar{x}(t), \bar{u}(t))}}_{B(t) \in \mathbb{R}^{n \times m}} \delta u(t). \end{aligned}$$

and similarly for the output

$$\delta y(t) \approx \underbrace{\left.\frac{\partial g}{\partial x}\right|_{(\bar{x}(t), \bar{u}(t))}}_{C(t) \in \mathbb{R}^{p \times n}} \delta x(t) + \underbrace{\left.\frac{\partial g}{\partial u}\right|_{(\bar{x}(t), \bar{u}(t))}}_{D(t) \in \mathbb{R}^{p \times m}} \delta u(t).$$

Hence, we are left with

$$\delta \dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t) \tag{11.3}$$

$$\delta y(t) = C(t)\delta x(t) + D(t)\delta u(t). \tag{11.4}$$

Remark. If $(\bar{x}(t), \bar{u}(t)) = (x^*, u^*)$ we have $f(x^*, u^*) = 0$ and $A(t) = A$, $B(t) = B$, $C(t) = C$ and $D(t) = D$.

Remark. δx and δu describe a deviation from the equilibrium trajectory.

Example 1. Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + 1 \\ \dot{x}_2 &= \sin(x_1) \cdot x_2 + x_1 \cdot u. \end{aligned}$$

The linearization around a trajectory $(\bar{x}(t), \bar{u}(t))$ is

$$\delta \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \cos(\bar{x}_1(t)) \cdot \bar{x}_2(t) + \bar{u}(t) & \sin(\bar{x}_1(t)) \end{bmatrix} \delta x(t) + \begin{bmatrix} 0 \\ \bar{x}_1(t) \end{bmatrix} \delta u(t).$$

The linearization around the equilibrium $x_1^* = 0, x_2^* = -1, u^* = 4$ is

$$\delta \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \delta x(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \delta u(t).$$

11.3 Solution to the Linear ODE

Consider a general linear time-invariant system of the form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du.\end{aligned} \tag{11.5}$$

Note that this is also what we get after linearizing a non-linear TI system and dropping the δ in the previous notation.
The solution to the ODE reads

$$\begin{aligned}x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau \\ y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau + D \cdot u(t)\end{aligned}$$

where

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

11.4 Stability of Linear Systems

For linear time-invariant systems of the form

$$\dot{x}(t) = Ax(t),$$

$A \in \mathbb{R}^{n\times n}$, with solution

$$x(t) = e^{At}x(0).$$

The Jordan decomposition of A reads

$$A = TJT^{-1},$$

and therefore

$$x(t) = T \cdot e^{Jt} \cdot T^{-1} \cdot x(0).$$

If A is a diagonalizable matrix, J is the diagonal matrix of the eigenvalues of A and T is the collection of the corresponding eigenvectors, that is

$$J = \Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\} \quad \text{and} \quad T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

Then the solution $x(t)$ reads

$$\sum_{i=1}^{n} c_i \cdot v_i e^{\lambda_i t}$$

for some $c_i \in \mathbb{R}$ depending on the initial condition.
We distinguish three cases as $t \to \infty$:

$$\begin{cases} \mathrm{Re}(\lambda_i) < 0 & x(t) \to 0; \\ \mathrm{Re}(\lambda_i) > 0 & \|x(t)\| \to \infty; \\ \mathrm{Re}(\lambda_i) = 0 & \|x(t)\| \text{ remains bounded.} \end{cases}$$

Definition 1 (Stable subspace E_s)**.** The stable subspace is the span of the eigenvectors associated with eigenvalues with negative real part; that is

$$E_\mathrm{s} = \mathrm{span}(v_i) \quad \text{such that} \quad \mathrm{Re}(\lambda_i) < 0.$$

Definition 2 (Unstable subspace E_u)**.** The unstable subspace is the span of the eigenvectors associated with eigenvalues with positive real part; that is

$$E_\mathrm{u} = \mathrm{span}(v_i) \quad \text{such that} \quad \mathrm{Re}(\lambda_i) > 0.$$

Definition 3 (Center subspace E_c)**.** The stable subspace is the span of the eigenvectors associated with eigenvalues with zero real part; that is

$$E_\mathrm{c} = \mathrm{span}(v_i) \quad \text{such that} \quad \mathrm{Re}(\lambda_i) = 0.$$

Remark. Assuming the matrix A is diagonalizable, then

$$\mathbb{R}^n = E_\mathrm{s} \oplus E_\mathrm{u} \oplus E_\mathrm{c},$$

i.e. the three subspaces span $\mathbb{R}^n$.
The operator $\oplus$ denotes a binary operation known as *Minkowski sum* and it is defined as follows. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$ be two vector spaces, then $Z = X \oplus Y \subseteq \mathbb{R}^n$ is given by

$$Z = X \oplus Y = \{x + y \,|\, x \in X, y \in Y\}$$

Lyapunov indirect method

Lyapunov indirect method exploits the linearized system to characterize *local* stability of equilibrium of the original non-linear dynamical system.
Consider a general autonomous system $\dot{x}(t) = f(x(t))$ with an equilibrium x^*. Define $\delta x(t) = x(t) - x^*$, linearize and obtain

$$\delta\dot{x}(t) = A \cdot \delta x(t). \tag{11.6}$$

Assume now, that A is diagonalizable, that is to say that, for all eigenvalues of A, the algebraic and geometric multiplicity coincides, i.e. A has n mutually linearly independent eigenvectors v_i. Then the linearized system (11.6) predicts the behavior around x^* of the original nonlinear system as follows:

Linearized System		Nonlinear System
δx^* is ...		x^* is ...
stable if $E_\mathrm{u} = \emptyset$	$\Longrightarrow$	No conclusion is possible; we need to check higher order terms.
(globally) asymptotically stable if $E_\mathrm{c} = E_\mathrm{u} = \emptyset$	$\Longrightarrow$	locally asymptotically stable
unstable if $E_\mathrm{u} \neq \emptyset$	$\Longrightarrow$	unstable

If, instead, A is *not diagonalizable* and the eigenvalues with zero real part have geometric multiplicity smaller than the corresponding algebraic multiplicity, i.e. the eigenvectors are *not* linearly independent, then, the equilibrium of the linear system is unstable even if A has no eigenvalues with positive real part.

11.5 Lyapunov Theorem for Linear-Time-Invariant Systems

Definition 4 (Hurwitz Matrix)**.** A matrix $A \in \mathbb{R}^n$ is Hurwitz if and only if all of its eigeinvalues have negative real part, equivalently $E_\mathrm{u} = E_\mathrm{c} = \emptyset$.

Definition 5 (Positive Definite Matrix)**.** A symmetric matrix Q is positive definite (notation: $Q \succ 0$) if and only if all of its eigenvalues are strictly positive.

For a LTI system of the form (11.6) the below statements are equivalent

1. A is Hurwitz.
2. For all symmetric, positive definite matrices $Q = Q^\top \succ 0$ exists a symmetric, positive definite matrix $P = P^\top \succ 0$ such that
$$V(x) = x^\top P x \implies \dot{V}(x) = -x^\top Q x.$$
3. The equilibrium at $\delta x^* = 0$ is asymptotically stable.

Under these conditions, the relationship between P and Q is given by the Lyapunov Equation:
$$A^\top P + PA = -Q. \tag{11.7}$$

11.6 Tips

No tips for today's exercise ☺.

11.7 Example

Since you are close to the end of the semester and your engineering team goes soon on holiday, you decide to work with the financial team of SpaghETH to build predictions for the future. After rigorous studies in the customers' behavior you come up with a dynamical model for costumers' interest in SpaghETH, denoted by x_1, and the revenue, denoted by x_2. Note that x_1 and x_2 denotes deviations from a reference configuration. The nonlinear system is described by the following differential equations:

$$\begin{aligned}\dot{x}_1 &= 3x_1x_2^2 + x_2^3x_1^2 \\ \dot{x}_2 &= x_1^2x_2 + 4x_2^3 - 4x_2 - x_1^3x_2^2.\end{aligned}$$

1. Linearize the system around the equilibrium $x^* = \begin{bmatrix}0 & 0\end{bmatrix}^\top$ and find the matrix A.
2. What can you conclude about the stability properties of the origin?
3. Evaluate the stability of the origin using the Lyapunov function

$$V = \frac{1}{2}(x_1^2 + x_2^2).$$

Solution.

1. The linearization matrix A reads

$$
\begin{aligned}
A &= \left.\begin{bmatrix} \frac{\partial}{\partial x_1}(3x_1x_2^2 + x_2^3x_1^2) & \frac{\partial}{\partial x_2}(3x_1x_2^2 + x_2^3x_1^2) \\ \frac{\partial}{\partial x_1}(x_1^2x_2 + 4x_2^3 - 4x_2 - x_1^3x_2^2) & \frac{\partial}{\partial x_2}(x_1^2x_2 + 4x_2^3 - 4x_2 - x_1^3x_2^2) \end{bmatrix}\right|_{(0,0)} \\
&= \left.\begin{bmatrix} 3x_2^2 + 2x_1x_2^3 & 6x_1x_2 + 3x_2^2x_1^2 \\ 2x_1x_2 - 3x_1^2x_2^2 & x_1^2 + 12x_2^2 - 4 - 3x_1^2x_2^2 \end{bmatrix}\right|_{(0,0)} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}.
\end{aligned}
$$

2. The eigenvalues of matrix A are $\lambda_1 = 0$ and $\lambda_2 = -4$. Using the Lyapunov principle, we cannot evaluate the stability of the origin of the nonlinear system, since the linearized one is just stable around the equilibrium.

3. The total time derivative of the Lyapunov function reads

$$
\begin{aligned}
\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= x_1 \cdot (3x_1x_2^2 + x_2^3x_1^2) + x_2 \cdot (x_1^2x_2 + 4x_2^3 - 4x_2 - x_1^3x_2^2) \\
&= 3x_1^2x_2^2 + x_2^2x_1^2 + 4x_2^4 - 4x_2^2 \\
&= 4x_2^2 \cdot (x_1^2 + x_2^2 - 1).
\end{aligned}
$$

   Note that $\dot{V} \leq 0$ for $x_1^2 + x_2^2 \leq 1$. That is, $\dot{V}(x) \leq 0$ for $\|x\|_2 \leq h$ for $h = 1$. Hence, we can conclude that the origin is stable.

   Remark. Note that $\dot{V}(x)$ is *not* negative definite. Note for instance that $\dot{V}(x) = 0$ for $x = \begin{bmatrix}1 & 0\end{bmatrix}^\top$, which clearly violates the condition $V(x) < 0$ for $x \neq 0$.

   Remark. Note that this does *not* mean that the origin is not (locally) asymptotically stable. A different Lyapunov function might succeed in proving (local) asymptotic stability of the equilibrium.