Exercise 9 - Parameter Identification

9.1 Introduction

So far, we have investigated how to come up with a mathematical model of a system. This modeling was based on laws from the field of physics, whereby we assumed that some parameters of the system were known. Now, these parameters are to be identified by running experiments.
It turns out that these experiments are usually not straightforward to perform since the correct input signals (test signals) have to be applied, i.e. the obtained data has to fully characterize the system's properties.
Once the data has been collected, it may be used for the following purposes:

- to identify unknown system structures and system parameters;
- to validate the results of the system modeling and parameter identification.

It is very important not to use the same data set for both purposes since doing so would result in an improper validation process.

9.2 Linear Least Squares Method

The linear least squares technique allows us to find the "best" set of parameters to fit a certain static, linear (in the parameters) model of the form

$$y(k) = h(u(k))^\top \cdot \pi + e(k) \qquad k = 1, 2, \ldots, r, \tag{9.1}$$

where

- $r \in \mathbb{N}^+$: total number of measurements;
- $y(k) \in \mathbb{R}$: system's output, k-th measurement;
- $u(k) \in \mathbb{R}^m$ system's input corresponding to the k-th measurement
- $h(\cdot) \in \mathbb{R}^q$: "regressor", describes the nonlinear dependence of the output from the input;
- $\pi \in \mathbb{R}^q$: vector of the q unknown system's parameters;
- $e(k) \in \mathbb{R}$: error of the k-th measurement.

Example 1. The model

$$y(k) = au(k)^2 + b + e(k)$$

is linear in the parameters. Let $\pi = \begin{bmatrix} a & b \end{bmatrix}^\top$, then the system can be brought in the form (9.1) with

$$y(k) = \underbrace{\begin{bmatrix} u(k)^2 & 1 \end{bmatrix}}_{h(u(k))^\top} \begin{bmatrix} a \\ b \end{bmatrix} + e(k).$$

This material is based on the HS17 teaching assistance taught by Nicolas Lanzetti and Gioele Zardini.
This document can be downloaded at `https://n.ethz.ch/~lnicolas`

Example 2. The model

$$y(k) = u(k)^a + au(k)$$

is nonlinear in the parameter a.

Example 3. The model

$$y(k) = u(k)^a$$

is nonlinear in the parameter a. However, if we write

$$\ln(y(k)) = \ln(u(k)^a) = a\ln(u(k))$$

the model is linear and of the form of Equation (9.1).

The least squares problem consists of finding the parameter vector π_{LS} that, given a data set, minimizes the sum of the errors as follows

$$\pi_{\mathrm{LS}} = \arg\min_{\pi} \sum_{k=1}^{r} e(k)^2 = \arg\min_{\pi} \sum_{k=1}^{r} (y(k) - h(u(k))^\top \pi)^2$$

Now let

$$e = \begin{bmatrix} e(1) \\ \vdots \\ e(r) \end{bmatrix} \qquad y = \begin{bmatrix} y(1) \\ \vdots \\ y(r) \end{bmatrix} \qquad H = \begin{bmatrix} h(u(1))^\top \\ \vdots \\ h(u(r))^\top \end{bmatrix}$$

then

$$\pi_{\mathrm{LS}} = \arg\min_{\pi} e^\top e = \arg\min_{\pi} (y - H\pi)^\top (y - H\pi).$$

The minimum can be obtained by setting the derivative w.r.t. π to zero[1]

$$\frac{\partial}{\partial \pi}\left(y^\top y - y^\top H\pi - \pi^\top H^\top y + \pi^\top H^\top H\pi\right) = -2H^\top y + 2H^\top H\pi \overset{!}{=} 0$$

which gives

$$\pi_{\mathrm{LS}} = (H^\top H)^{-1} H^\top y. \tag{9.2}$$

Remark. The matrix $H^\dagger = (H^\top H)^{-1}H^\top$ is also called Moore-Penrose inverse of H. In MATLAB `pinv(H)`.

Remark. In MATLAB, (9.2) is implemented using `(H'*H)\H'*y`. Explicitly computing the inverse of a matrix (as in `(H'*H)^(-1)`) may result in numerical instabilities.

Weighted Least Squares Method

Assume now that the errors are weighted by some weigths $w(k) > 0$. Then,

$$\pi_{\mathrm{LS}} = \arg\min_{\pi} \sum_{k=1}^{r} w(k)e(k)^2 = \arg\min_{\pi} \sum_{k=1}^{r} w(k)(y(k) - h(u(k))^\top \pi)^2$$

[1] Stricly speaking, it should be shown that π_{LS} is a minimum. However, as the matrix of second derivatives $2H^\top H$ is positive definite, π_{LS} a minimum.

Using the same notation as above and with $W = \text{diag}(w_1, \ldots, w_r)$ we obtain

$$\pi_{\text{LS}} = \arg\min_{\pi} e^\top W e = \arg\min_{\pi} (y - H\pi)^\top W (y - H\pi).$$

Setting the derivative to 0 gives

$$\pi_{\text{LS}} = (H^\top W H)^{-1} H^\top W y.$$

Remark. In a probabilistic environment, it might be reasonable to set $w(k) = 1/\sigma(k)^2$, where $\sigma(k)^2$ is the standard deviation of measurement k.

Remark. Non-diagonal matrices W might be used to penalize unlikely combinations of errors.

9.3 Iterative Least Squares Methods

In the previous sections we presented a batch approach, i.e., a methodology where data is first collected, organized, and then the solution is computed. However, this approach might be computationally demanding for real-time applications since at each time step the whole problem has to be solved again. Therefore we try to come up with an algorithm that allows us to recursively update π_{LS} as new measurements
are available. As shown in the lecture, a possible way is the following:

$$\begin{aligned}
\pi_{\text{LS}}(r+1) &= \pi_{\text{LS}}(r) + \frac{1}{1 + c(r+1)} \cdot \Omega(r) \cdot h(u(r+1)) \cdot \big(y(r+1) - h(u(r+1))^\top \pi_{\text{LS}}(r)\big) \\
\Omega(r+1) &= \Omega(r) - \frac{1}{1 + c(r+1)} \cdot \Omega(r) \cdot h(u(r+1)) \cdot h(u(r+1))^\top \cdot \Omega(r) \\
c(r+1) &= h(u(r+1))^\top \cdot \Omega(r) \cdot h(u(r+1)).
\end{aligned}$$

Remark. If we have $r > 0$, the recursive algorithm can be initialized by calculating $\pi_{\text{LS}}(r)$ with Equation (9.2) and setting

$$\Omega(r) = \sum_{k=1}^{r} h(u(k)) \cdot h(u(k))^\top,$$

whereas, in case no measurements are available, a possible (simple) initialization is

$$\pi_{\text{LS}}(0) = \text{E}[\pi], \qquad \Omega(0) = \mathbb{I}_{q \times q}.$$

There exist many extensions/variations for the recursive least squares method. Among them, we highlight:

- The least squares with exponential forgetting, where old data is weighted with some forgetting factor $\lambda \in (0, 1)$. That is, at time k the measurement has weight λ^{r-k}. Then,

$$\pi_{\text{LS},\lambda} = \arg\min_{\pi} \sum_{k=1}^{r} \lambda^{r-k} e(k)^2 = \arg\min_{\pi} \sum_{k=1}^{r} \lambda^{r-k} (y(k) - h(u(k))^\top \pi)^2$$

- The Kaczmarz's projection algorithm, which is more computationally efficient than the regular least squares algorithm.

9.4 Identification of Parameters in Dynamic Systems

So far, we assume that the model of the system was algebraic, i.e.,

$$y(k) = h(u(k))^\top u(k) + e(k).$$

If we have systems of the form

$$\pi_0 \frac{\mathrm{d}}{\mathrm{d}t} y(t) = h(u(t))\pi + e(t).$$

how should we proceed?
We distinguish between two cases:

- Taking measurements in steady state. That is, by considering

  $$0 = h(u(t))\pi + e(t)$$

  and doing least squares for all the available measurements. However, this does not allow us to estimate π_0 as $\frac{\mathrm{d}}{\mathrm{d}t} y = 0$.

- Numerically finding the optimal parameters. That is,

  $$\pi_0, \pi = \arg\min_{\pi_0,\pi} \sum_{k=1}^{n} (y(k) - \hat{y}(k))^2,$$

  where $\hat{y}$ is the output of the simulated system. In other words, we try to find the parameters that minimize the error between the output of the simulation and the measurements. Note that there exists no closed from equation of π_0 and π; the problem has to be tackled numerically.

9.5 Example

Since your SpaghETH is going well, you want to optimize and control the production of your meals depending on how satisfied your customers are. In order to analyze this, you find a model to describe the satisfaction of your customers at any time t. This could help you standardizing your whole production process and designing a controller which can handle any situation.
The model is mathematically given as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S(t) &= T\cdot q_{\text{meal}} + D\cdot\alpha_{\text{meal}} - S(t)\\ \frac{T}{C} &= P.\end{aligned}$$

The first equation of the model relates the satisfaction $S(t)$ of the cooked meal to several factors. Namely, this depends on

- the percentage of waiting time T for the customers to receive their meal ($T = \frac{\text{waiting time}}{\text{total permanence time}}$),
- the quality of the meal q_{meal},
- the digestibility of the meal D,
- a factor describing the size of the portion α_{meal},
- and the satisfaction itself $S(t)$.

The second equation, defines the waiting time percentage for receiving the meal T, as a function of

- the normalized number of pots needed to cook all the meals P, i.e. the sum of employed pots divided by the number of pots normally used to cook a meal,
- and of the busy ratio of the food truck C, i.e. the ratio of customers currently served to the maximum customers that can be served.

We are interested in finding the parameters T and α_{meal} with the Least Squares Method, using data from the feedbacks received at the end of the stay (that is to say in steady state) on two meals: pasta al ragù and Älplermagronen.

	C	D	S	P	q_{meal}
Pasta al ragù (R)	0.5	3	4	1	1
Älplermagronen (A)	1	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{3}$

1. Write down the least squares problem for the steady state of the system.
2. Determine the parameters T and α_{meal} with the least squares method.
3. By computing π_{LS} you realize that something does not work. You find out that the intern who should count the number of pots for Älplermagronen was drunk and thus the number he reported is not reliable. How can you take this into account within the framework provided by the least squares method? Provide a possible solution.

4. You realize that the waiting time T is described better as

$$(P \cdot C)^T = \beta,$$

where $0 < \beta < 1 \in \mathbb{R}$ is a known constant. Solve the Least Squares Problem (only for the model of the waiting time) for the following measurements.

	C	P
Pasta al ragù (R)	0.5	1
Älplermagronen (A)	1	$\sqrt{3}$

Solution.

1. We consider the system in its equilibrium state. The relations become

$$S_i = T \cdot q_{\text{meal},i} + D_i \cdot \alpha_{\text{meal}}$$
$$P_i = \frac{T}{C_i},$$

where $i \in \{R, A\}$. The parameter vector we look for reads

$$\pi = \begin{bmatrix} T \\ \alpha_{\text{meal}} \end{bmatrix}.$$

The measurement vector $\tilde{y}$ reads

$$\tilde{y} = \begin{bmatrix} S_R \\ P_R \\ S_A \\ P_A \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \\ \sqrt{3} \end{bmatrix}.$$

and the H matrix reads

$$H = \begin{bmatrix} q_{\text{meal,R}} & D_{\text{R}} \\ \frac{1}{C_{\text{R}}} & 0 \\ q_{\text{meal,A}} & D_{\text{A}} \\ \frac{1}{C_{\text{A}}} & 0 \end{bmatrix}.$$

2. Using the general solution for the least square problem (see Eq. (9.2)), one obtains π_{LS} as

$$\begin{aligned} \pi_{\text{LS}} &= \left(H^\top H\right)^{-1} H^\top \tilde{y} \\ &= \left(\begin{bmatrix} 1 & 2 & \sqrt{3} & 1 \\ 3 & 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ \sqrt{3} & \sqrt{3} \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & \sqrt{3} & 1 \\ 3 & 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \\ \sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} 9 & 6 \\ 6 & 12 \end{bmatrix}^{-1} \begin{bmatrix} 6+3\sqrt{3} \\ 12+2\sqrt{3} \end{bmatrix} \\ &= \frac{1}{72} \begin{bmatrix} 12 & -6 \\ -6 & 9 \end{bmatrix} \begin{bmatrix} 6+3\sqrt{3} \\ 12+2\sqrt{3} \end{bmatrix} \\ &= \frac{1}{72} \begin{bmatrix} 72+36\sqrt{3}-72-12\sqrt{3} \\ -36-18\sqrt{3}+108+18\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{3}}{3} \\ 1 \end{bmatrix} \end{aligned}$$

3. In order to take into account the reduced confidence in one of the measurement we make use of the weighted least square method. We (arbitrarily) opt to weigh the faulty information 1/10 of the other measurements. Thus, W reads

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}.$$

The parameter vector $\pi_{\text{LS,W}}$ can be computed using the weighted least square formula

$$\begin{aligned}
\pi_{\text{LS,W}} &= \left(H^\top W H\right)^{-1} H^\top W \tilde{y} \\
&= \left(\begin{bmatrix} 1 & 2 & \sqrt{3} & 1 \\ 3 & 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ \sqrt{3} & \sqrt{3} \\ 1 & 0 \end{bmatrix} \right)^{-1} \cdot \\
&\quad \begin{bmatrix} 1 & 2 & \sqrt{3} & 1 \\ 3 & 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \\ \sqrt{3} \end{bmatrix} \\
&= \cdots \\
&= \begin{bmatrix} \frac{11\sqrt{3}}{51} \\ 1 + \frac{\sqrt{3}}{17} \end{bmatrix}
\end{aligned}$$

Remark. We could have chosen

$$\tilde{W} = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

without changing the final result. The only thing that matters is the *relative* weighing.

4. Since the model is not linear, we cannot apply the standard method directly. However, with a simple transformation is possible to obtain a linear model, namely

$$(P \cdot C)^T = \beta \qquad \Longleftrightarrow \qquad \ln\left((P \cdot C)^T\right) = \ln(\beta)$$

and we hence obtain a linear model by noticing that

$$T \ln(P \cdot C) = \ln(\beta).$$

Within this setting we have

$$\begin{aligned}
\pi_{\text{LS}} &= \begin{bmatrix} T \end{bmatrix}, \\
\tilde{y} &= \begin{bmatrix} \ln(\beta) \\ \ln(\beta) \end{bmatrix}, \\
H &= \begin{bmatrix} \ln\left(P_{\text{R}} \cdot C_{\text{R}}\right) \\ \ln\left(P_{\text{A}} \cdot C_{\text{A}}\right) \end{bmatrix} = \begin{bmatrix} \ln(0.5) \\ \ln\left(\sqrt{3}\right) \end{bmatrix}, \\
W &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Finally, it holds

$$\begin{aligned}\pi_{\mathrm{LS,W}} &= \left(H^\top W H\right)^{-1} H^\top W \tilde{y} \\ &= \left(\begin{bmatrix}\ln(0.5) & \ln(\sqrt{3})\end{bmatrix}\begin{bmatrix}\ln(0.5) \\ \ln(\sqrt{3})\end{bmatrix}\right)^{-1}\begin{bmatrix}\ln(0.5) & \ln(\sqrt{3})\end{bmatrix}\begin{bmatrix}\ln(\beta) \\ \ln(\beta)\end{bmatrix} \\ &= \cdots \\ &= -\frac{4\ln\left(\frac{2\sqrt{3}}{3}\right)\ln\left(\beta\right)}{4\ln\left(2\right)^2 + \ln\left(3\right)^2}.\end{aligned}$$