Exercise 7 - Fluiddynamic Systems

7.1 Valves

The flow of fluids between reservoirs is determined by *valves*, whose inputs are the pressure up- and downstream, denoted by p_{in} and p_{out} respectively. Here, we present methods to model valves with

- incompressible fluids and
- compressible fluids.

7.1.1 Modeling Assumptions

- The friction effects in the flow are modeled by using experimentally determined correction factors.
- Inertial effects in the flow are neglected. In fact, the mass of fluid around the valve is very small if compared to the mass stored in the receiving reservoirs.
- The valves are insulated.
- All flow phenomena are *zero dimensional*, i.e., spatial dependences are neglected.

7.1.2 Valves with Incompressible Fluids

Incompressible fluids (constant density) are characterized by low *Mach* numbers, where the Mach number is defined as

$$M = \frac{u}{c},$$

where u is local flow velocity and c the local speed of sound through the medium.

Remark. As a rule of thumb, we can consider a fluid to be incompressible if $M < 0.3$ and the flow is quasi-steady and isothermal.

The fluid mass flow $\overset{*}{m}$ can be then derived by using a modified Bernoulli equation. This reads

$$\overset{*}{m}(t) = c_{\text{d}} \cdot A \cdot \sqrt{2 \cdot \rho \cdot (p_{\text{in}}(t) - p_{\text{out}}(t))}, \tag{7.1}$$

where c_{d} is the discharge coefficient (factor which takes into account flow restrictions, friction and other losses), A is the open area of the valve and ρ is the density of the fluid.

This material is based on the HS17 teaching assistance taught by Nicolas Lanzetti and Gioele Zardini.
This document can be downloaded at `https://n.ethz.ch/~lnicolas`

7.1.3 Valves with Compressible Fluids

The model we use to deal with compressible effects is the *isenthalpic throttle*. The name originates from the fact that the fluid enthalpy crossing the valve remains constant, i.e. the process is adiabatic and no work is performed on the system.

By referring to Figure 1, one can divide the valve into two distinct regions (before and after the vertical line respectively).

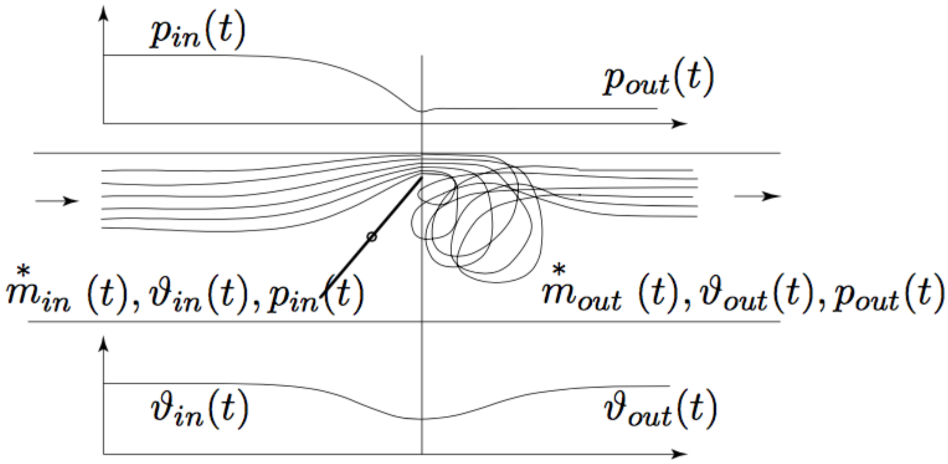

Figure 1: Two regions for the model.

Upstream Region: isentropic conversion of pressure in kinetic energy, i.e. flow velocity increases (pressure decreases); laminar flow.

Downstream Region: fluid decelerates, kinetic energy is converted in thermal energy and no (or little) pressure recuperation occurs; turbolent flow.

Furthermore we can see that

- the pressure in the narrowest part of the valve is approximately equal to the downstream pressure.
- The temperature of the flow before and after the valve is approximately the same.

Using the first law of thermodynamics and the properties of isentropic expansions for a perfect gas, we can develop an equation for the mass flow

$$\overset{*}{m}(t) = c_{\mathrm{d}} \cdot A(t) \cdot \frac{p_{\mathrm{in}}(t)}{\sqrt{R \cdot \vartheta_{\mathrm{in}}}} \cdot \Psi(p_{\mathrm{in}}(t), p_{\mathrm{out}}(t)),$$

where

$$\Psi(p_{\mathrm{in}}(t), p_{\mathrm{out}}(t)) = \begin{cases} \sqrt{\gamma \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} & \text{for } p_{\mathrm{out}}(t) < p_{\mathrm{cr}}(t), \\ \left(\frac{p_{\mathrm{out}}(t)}{p_{\mathrm{in}}(t)}\right)^{\frac{1}{\gamma}} \cdot \sqrt{\frac{2\gamma}{\gamma-1} \cdot \left[1 - \left(\frac{p_{\mathrm{out}}(t)}{p_{\mathrm{in}}(t)}\right)^{\frac{\gamma-1}{\gamma}}\right]} & \text{for } p_{\mathrm{out}}(t) \geq p_{\mathrm{cr}}(t), \end{cases}$$

and

$$p_{\mathrm{cr}} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \cdot p_{\mathrm{in}}(t),$$

where γ is the specific heat ratio.

At critical pressure p_{cr}, the flow reaches in its narrowest part sonic conditions, i.e. $M = 1$. For air and similar gases a simplification can be taken into account. This reads

$$\Psi(p_{\mathrm{in}}(t), p_{\mathrm{out}}(t)) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } p_{\mathrm{out}} < 0.5 p_{\mathrm{in}}, \\ \sqrt{\frac{2p_{\mathrm{out}}}{p_{\mathrm{in}}} \cdot \left[1 - \frac{p_{\mathrm{out}}}{p_{\mathrm{in}}}\right]} & \text{for } p_{\mathrm{out}} \geq 0.5 p_{\mathrm{in}}. \end{cases}$$

7.2 Tips

Exercise (a): Don't forget to include the discharge coefficients for the respective inlets and outlets.

Exercise (b): Model the friction forces as dampers with coefficient γ.

Exercise (c): No tips.

Exercise (d): No tips.

7.3 Example

As SpaghETH is rapidly growing, you decide to buy a building where pasta, risotto, and sauces can be easily stored. Due to the large orders, your chief technical manager suggests to study a way to efficiently move around the heavy boxes. You opt for a spring-fluiddynamic system, consisting of a chamber, two valves, a piston, and a box, as depicted in Figure 2.
The chamber, modeled as a cylinder of diameter D, is connected to a reservoir of air with constant pressure $p_{\mathrm{r}} = 10\,\mathrm{bar}$ and constant temperature ϑ_{r} through a valve of opening area $A_1(t)$ and discharge coefficient $c_{\mathrm{d},1}$. Moreover, the chamber is connected to the ambient, where the pressure and temperature are $p_\infty = 1\,\mathrm{bar}$ and ϑ_∞, through second valve with opening $A_2(t)$ and discharge coefficient $c_{\mathrm{d},2}$. Experiments have shown that the pressure in the chamber changes dynamically and is (on average) approximately 1.5 bar. To model the valves you may assume constant $\gamma = 1.4$ and constant specific heats c_v and c_p. You use simplified models. The walls of the chamber have the constant temperature ϑ_∞ and the heat transfer coefficient for the internal wall is α. No heat is transferred to the piston. The piston and the box have mass m_{p} and mass m_{b}, respectively. They move frictionless on a flat surface. The spring has the constant k and is unstretched for $x = 0$.

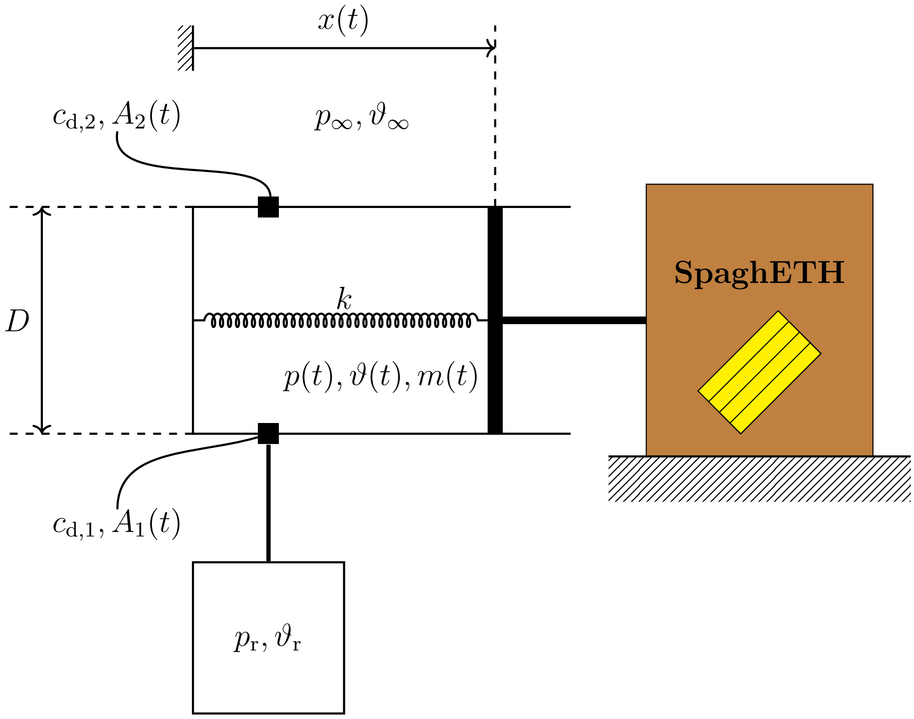

Figure 2: Sketch of the system.

1. What are the input(s) and output(s) to the system.
2. List the reservoirs with the level variables.
3. Draw a causality diagram of the system.
4. In what conditions will the valves operate?
5. Formulate the relations describing each block.

Solution.

1. The inputs are the opening surfaces of the two valves. The output is the position of the piston (or of the box).

2. The reservoirs of the system are:

   - Mass of air in the chamber with level variable $m(t)$;
   - Internal energy in the chamber with level variable $\vartheta(t)$;
   - Kinetic energy of the piston and of the box with level variable $\dot{x}(t)$;
   - Potential energy in the spring with level variable $x(t)$.

3. The causality diagram is sketched in Figure 3.

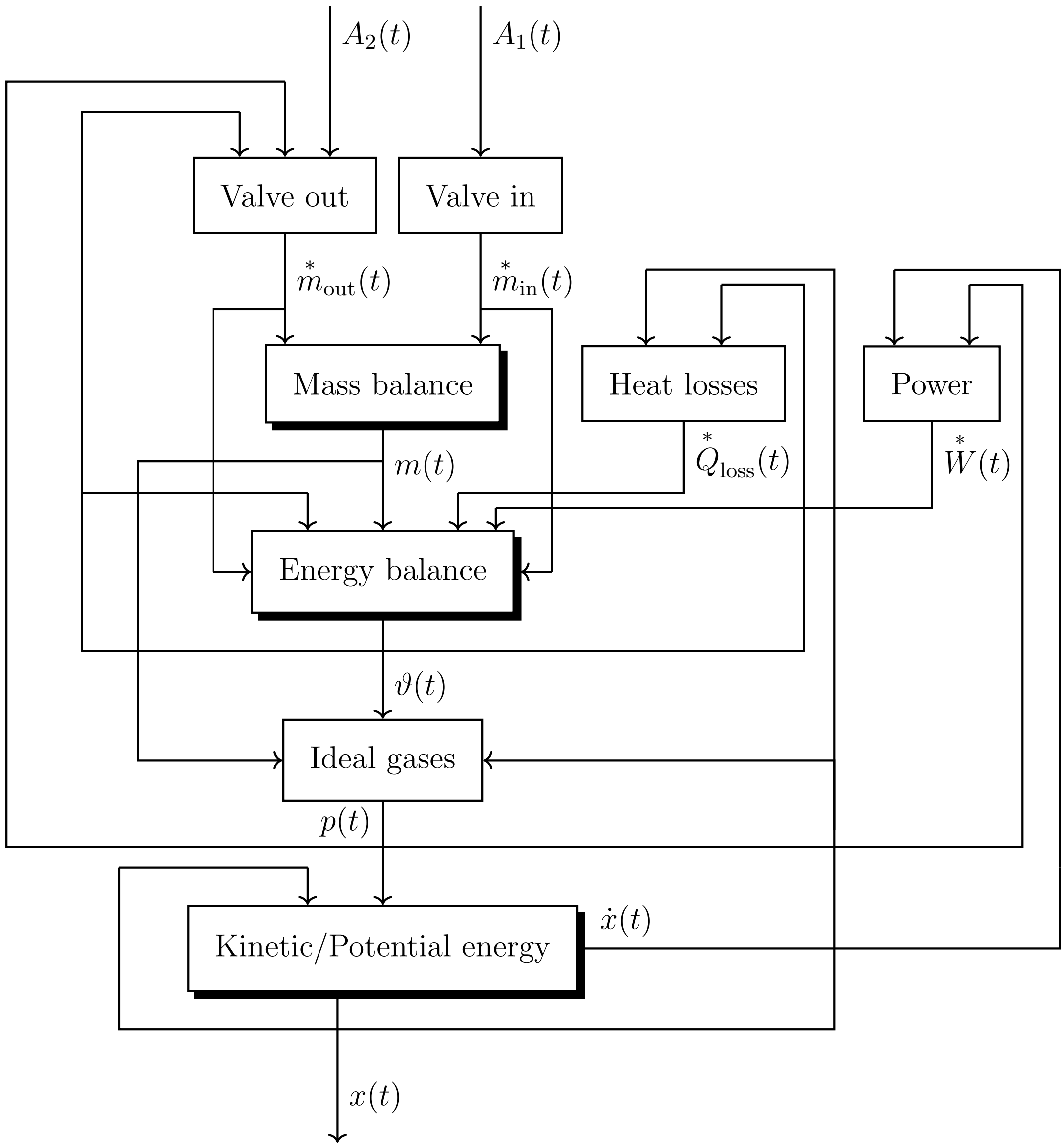

Figure 3: Causality diagram of the system.

4. For the first valve we have

$$p_{\text{cr}} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \cdot 10\,\text{bar} \approx 5.3\,\text{bar} > p_{\text{out}}(t).$$

Hence, the valve operates in "sonic conditions". For the second valve we have

$$p_{\text{cr}} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \cdot 1.5\,\text{bar} \approx 0.8\,\text{bar} < p_{\text{out}}(t).$$

Hence, the valve operates in "normal conditions".

5. The relations for the valves are

$$\overset{*}{m}_{\text{in}}(t) = c_{\text{d},1} \cdot A_1(t) \cdot \frac{p_{\text{r}}}{\sqrt{R \cdot \vartheta_{\text{r}}}} \cdot \sqrt{\gamma \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

$$\overset{*}{m}_{\text{out}}(t) = c_{\text{d},2} \cdot A_2(t) \cdot \frac{p(t)}{\sqrt{R \cdot \vartheta(t)}} \cdot \left(\frac{p_\infty}{p(t)}\right)^{\frac{1}{\gamma}} \cdot \sqrt{\frac{2\gamma}{\gamma-1} \cdot \left[1 - \left(\frac{p_\infty}{p(t)}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

The mass balance yields then

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) = \overset{*}{m}_{\text{in}}(t) - \overset{*}{m}_{\text{out}}(t).$$

The energy balance reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c_v \cdot m(t) \cdot \vartheta(t)\right) = \overset{*}{m}_{\text{in}}(t) \cdot c_p \cdot \vartheta_{\text{r}} - \overset{*}{m}_{\text{out}}(t) \cdot c_p \cdot \vartheta(t) - \overset{*}{W}(t) - \overset{*}{Q}_{\text{loss}}(t),$$

where

$$\overset{*}{Q}_{\text{loss}}(t) = \alpha \cdot (S + \pi D \cdot x(t)) \cdot (\vartheta(t) - \vartheta_\infty),$$
$$\overset{*}{W}(t) = p(t) \cdot S \cdot \dot{x}(t)$$

with $S = \pi D^2/4$. Note that the pressure can then be computed by using the ideal gas relation

$$p(t) = \frac{1}{V(t)} \cdot m(t) \cdot R \cdot \vartheta(t).$$

where $V(t) = S \cdot x(t)$ and $R = c_p - c_v$. The position of the piston obeys to

$$(m_{\text{p}} + m_{\text{b}}) \cdot \frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) = (p(t) - p_\infty) \cdot S - k \cdot x(t).$$

Remark. In an earlier version of this document, it was wrongly stated that the work on the gas $\overset{*}{W}$ is computed as $\overset{*}{W} = (p(t) - p_\infty) \cdot S \cdot \dot{x}(t)$. Even though the *net work on the piston* is indeed computed using the pressure difference, the work exchange between the system and the environment is *independent of the ambient pressure.*