Exercise 6 - Thermodynamic Systems

The contents provided in this document provide a general overview of the tools needed to model thermodynamic systems. For a more in depth treatment of this material, please refer to *Fundamentals of Engineering Thermodynamics, M.J. Moran and H.N. Shapiro.*

6.1 Internal Energy

The variation of internal energy for a **closed system** with mass m and uniform temperature ϑ is[1]

$$\mathrm{d}U = \delta W + \delta Q,$$

or equivalently

$$\Delta U = W + Q, \tag{6.1}$$

where

- U is the internal energy,
- W is the work of external forces onto the system,

$$\delta W = -p_{\text{ext}} \cdot \mathrm{d}V,$$

- and Q is the thermal energy exchanged with the surrounding.

The internal energy can be computed *at constant volume* as

$$U(\vartheta) = \int_{\vartheta_0}^{\vartheta} m \cdot c_v(\vartheta)\,\mathrm{d}\vartheta \approx m \cdot c_v \cdot (\vartheta - \vartheta_0)$$

for approximately constant c_v. We distinguish between three special cases:

- Adiabatic process: no heat transfer, $Q = 0$, $\Delta U = W$.
- Isochoric process: no volume change, $W = 0$, $\Delta U = Q$.
- Isolated systems: $\Delta U = 0$.

Remark. We recall the following:

- These assumptions are also valid for compressible gases if temperature variations are not too large.
- We normally consider $U = 0$ at 0 Kelvin: i.e., $\vartheta_0 = 0\,\mathrm{K}$.

[1] Note that δ is used to indicate an inexact differential, a concept used to indicate the path dependence of the quantity of interest. Furthermore, this formulation of the energy balance neglects changes in kinetic and potential energy.

This material is based on the HS17 teaching assistance taught by Nicolas Lanzetti and Gioele Zardini.
This document can be downloaded at https://n.ethz.ch/~lnicolas

6.2 Enthalpy

The enthalpy is defined as

$$H = U + pV$$

and can be computed *at constant pressure* as

$$H(\vartheta) = \int_{\vartheta_0}^{\vartheta} m \cdot c_p(\vartheta)\,\mathrm{d}\vartheta \approx m \cdot c_p \cdot (\vartheta - \vartheta_0)$$

for approximately constant c_p. We can now formulate the energy balance for a open system:

$$\frac{\mathrm{d}}{\mathrm{d}t}U = \sum \overset{*}{H}_{\text{in}} - \sum \overset{*}{H}_{\text{out}} + \overset{*}{Q} + \overset{*}{W}, \tag{6.2}$$

where

- U is the internal energy,
- $\overset{*}{H} = \overset{*}{m} \cdot c_p \cdot \vartheta$ is the enthalpy flow associated with mass transfer,
- $\overset{*}{W}$ is the mechanical power applied to the system,
- and $\overset{*}{Q}$ is the heat flow exchanged with the surroundings.

6.3 Incompressible Bodies

For the special case of incompressible bodies we have that the specific heats are equal; that is, $c = c_p = c_v$. With the additional assumption that they are constant we get

$$U(\vartheta) = H(\vartheta) = m \cdot c \cdot (\vartheta - \vartheta_0).$$

6.4 Ideal Gases

Recall that for an ideal gas we have

$$pV = n\bar{R}\vartheta = mR\vartheta,$$

where $\bar{R} = 8.134\,{}^{\mathrm{J}}/_{\mathrm{mol\cdot K}}$ is the universal gas constant and $R = \bar{R}/M_{\text{gas}}$, where M_{gas} is the molar mass of the gas. For ideal gases, both the internal energy and the enthalpy are a function of the temperature alone, i.e.,

$$\begin{aligned} H(\vartheta) &= m \cdot c_p \cdot (\vartheta - \vartheta_0), \\ U(\vartheta) &= m \cdot c_v \cdot (\vartheta - \vartheta_0), \end{aligned}$$

even if volume and/or pressure are not constant. Moreover, the following relations hold

$$\begin{aligned} R &= c_p - c_v, \\ \gamma &= \frac{c_p}{c_v} > 1. \qquad \text{(heat capacity ratio)} \end{aligned}$$

6.5 Heat Transfer

Essentially, there are three ways in which heat transfer happens naturally.

Heat Conduction: which is described by Fourier's law. For the one-dimensional case we have

$$\overset{*}{Q} = \frac{\kappa A}{l}(\vartheta_1 - \vartheta_2), \qquad [\kappa] = \frac{\mathrm{W}}{\mathrm{m} \cdot \mathrm{K}}$$

where κ is the thermal conductivity, A is the cross-sectional area, and l is the length of the medium.

Heat Convection: which is described by Newton's law. The heat transfer between a solid body with contact surface A and temperature ϑ_1 and the surrounding fluid with temperature ϑ_2 is

$$\overset{*}{Q} = kA(\vartheta_1 - \vartheta_2), \qquad [k] = \frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{K}}$$

where k is the heat transfer coefficient.

Heat Radiation: which is described by Stefan-Bolzmann's law. The heat radiation from a body with surface A and temperature ϑ_1 to the surrounding at temperature ϑ_2 is

$$\overset{*}{Q} = \varepsilon\sigma A(\vartheta_1^4 - \vartheta_2^4),$$

where $\varepsilon \in [0, 1]$ is the emissivity of the body and $\sigma = 5.670 \cdot 10^{-8}\, \mathrm{W}/\mathrm{m}^2 \cdot \mathrm{K}^4$ is the Stefan-Bolzmann constant.

6.6 Tips

Exercise 1: Modeling the mass flow through an isenthalpic throttle is a topic of next week's lecture, use the given function blindly.

Exercise 2: The heat flow provided by the combustion is given by

$$\overset{*}{Q}_{\text{burn}}(t) = H_l \cdot \overset{*}{m}_g(t).$$

6.7 Example

In order to extend your businness and attract new customers to your SpaghETH, you decide to recreate the typical "Ticino Experience" by cooking the real Risotto[2] and serving it on the Polyterrasse. Everybody knows (or should know!) that a keypoint for a good Risotto is the so called *Mantecatura*. This ancient procedure consists in mixing butter and cheese (any kind of cheese is ok, the more the better) with the hot rice.
We model the hot rice mixture (i.e., rice, melted butter and melted cheese) as a unique element with mass m_r. Here, we opt for a traditional recipe, using butter and Parmigiano Reggiano. Both the butter and the Parmigiano Reggiano can be modeled to have a cubic form with mass m_b and m_p and **constant** temperature ϑ_b and ϑ_p, respectively. The hot rice mixture is continuously mixed with a wooden spoon providing a given power $\overset{*}{W}_\mathrm{stir}$. While you are "mantecating", your flatmate throws in some peperonata with specific heat c_pep and temperature ϑ_pep into the pot at rate $\overset{*}{m}_\mathrm{pep}$. Since the pot is assumed to be well isolated, the heat exchange occurs only with the air at the top of the pot via convection (area A_r, temperature ϑ_air, and heat transfer coefficient α_r). The whole process occurs at ambient pressure. A sketch of the system is depicted in Figure 1. For the modeling of the system, consider the following assumptions:

- Only one cube of butter and one cube of Parmigiano are put in the pot.
- The melting enthalpy and the melting temperature of the butter are L_b and $\vartheta_\mathrm{m,b}$. The specific heat is c_b and the density is ρ_b. The heat exchange coefficient with the hot rice is α_b.
- The melting enthalpy and the melting temperature of the Parmigiano Reggiano are L_p and $\vartheta_\mathrm{m,p}$. The specific heat is c_p and the density is ρ_p. The heat exchange coefficient with the hot rice is α_p.

1. List all the reservoirs with the corresponding level variables of the system.
2. Compute the mass-flow of melting butter.
3. Compute the mass-flow of melting Parmigiano Reggiano.
4. Formulate a differential equation for the mass and one for the temperature of the hot rice mixture.

[2]Note that in Ticino dialect one says "*Ul Risott Ticines*".

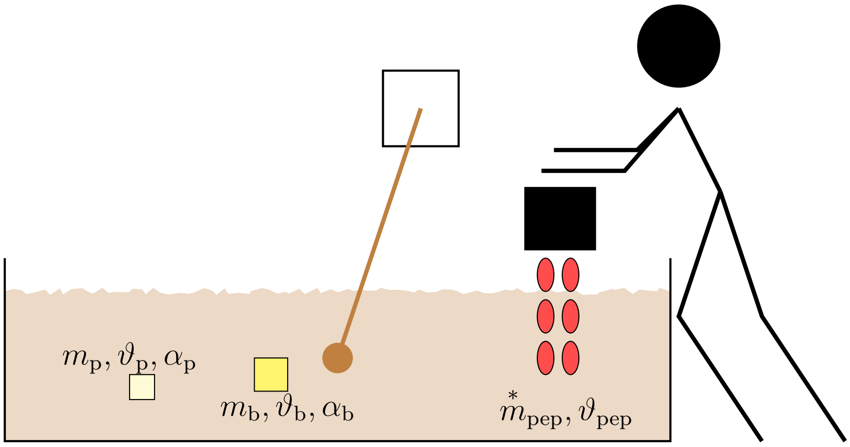

Figure 1: Sketch of the system.

Solution.

1. The reservoirs of the system are:

   - The mass of the butter m_b, whose level variable is the mass itself.
   - The mass of the parmigiano m_p, whose level variable is the mass itself.
   - The mass of the hot rice mixture m_r, whose level variable is the mass itself.
   - The internal energy of the hot rice U_r, whose level variable is the temperature $\vartheta_\mathrm{r}(t)$.

2. A mass balance on the butter gives:

$$\frac{\mathrm{d}}{\mathrm{d}t} m_\mathrm{b}(t) = -\overset{*}{m}_\mathrm{b}(t),$$

where $\overset{*}{m}_\mathrm{b}$ is the melting butter mass flow. The energy balance reads

$$\frac{\mathrm{d}}{\mathrm{d}t} U_\mathrm{b}(t) = -\overset{*}{m}_\mathrm{b}(t) \cdot h_\mathrm{b} + \overset{*}{Q}(t),$$

which reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(m_\mathrm{b}(t) \cdot c_\mathrm{b} \cdot \vartheta_\mathrm{b}\right) = -\overset{*}{m}_\mathrm{b}(t) \cdot (L_\mathrm{b} + c_\mathrm{b} \cdot \vartheta_\mathrm{m,b}) + \alpha_\mathrm{b} \cdot A_\mathrm{b}(t) \cdot (\vartheta_\mathrm{r}(t) - \vartheta_\mathrm{b}).$$

As ϑ_b is constant and $\frac{\mathrm{d}}{\mathrm{d}t} m_\mathrm{b} = -\overset{*}{m}_\mathrm{b}$ the above equation simplifies to

$$-\overset{*}{m}_\mathrm{b}(t) \cdot c_\mathrm{b} \cdot \vartheta_\mathrm{b} = -\overset{*}{m}_\mathrm{b}(t) \cdot (L_\mathrm{b} + c_\mathrm{b} \cdot \vartheta_\mathrm{m,b}) + \alpha_\mathrm{b} \cdot A_\mathrm{b}(t) \cdot (\vartheta_\mathrm{r}(t) - \vartheta_\mathrm{b}).$$

Basic algebraic manipulations lead to

$$\overset{*}{m}_\mathrm{b}(t) = \frac{\alpha_\mathrm{b} \cdot A_\mathrm{b}(t)}{L_\mathrm{b} + c_\mathrm{b} \cdot (\vartheta_\mathrm{m,b} - \vartheta_\mathrm{b})} \cdot (\vartheta_\mathrm{r}(t) - \vartheta_\mathrm{b}),$$

where the butter surface $A_\mathrm{b}(t)$ depends on the remaining mass of butter and reads

$$A_\mathrm{b}(t) = 6 x_\mathrm{b}^2 = 6 \cdot \left(\frac{m_\mathrm{b}(t)}{\rho_\mathrm{b}}\right)^{\frac{2}{3}}.$$

3. Analogously, the melting parmigiano mass flow $\overset{*}{m}_\mathrm{p}$ can be computed as

$$\overset{*}{m}_\mathrm{p}(t) = \frac{\alpha_\mathrm{b} \cdot A_\mathrm{p}(t)}{L_\mathrm{p} + c_\mathrm{p} \cdot (\vartheta_\mathrm{m,p} - \vartheta_\mathrm{p})} \cdot (\vartheta_\mathrm{r}(t) - \vartheta_\mathrm{p}),$$

   where the parmigiano surface $A_\mathrm{p}(t)$ is

$$A_\mathrm{p}(t) = 6x_\mathrm{p}^2 = 6 \cdot \left(\frac{m_\mathrm{p}(t)}{\rho_\mathrm{p}}\right)^{\frac{2}{3}}.$$

4. The change in mass for the hot rice reads

$$\frac{\mathrm{d}}{\mathrm{d}t} m_\mathrm{r}(t) = \overset{*}{m}_\mathrm{pep} + \overset{*}{m}_\mathrm{b}(t) + \overset{*}{m}_\mathrm{p}(t).$$

   The change in internal energy of the hot rice can be computed with the first law of thermodynamics for open system as

$$\frac{\mathrm{d}}{\mathrm{d}t} U_\mathrm{r}(t) = \overset{*}{Q}(t) + \overset{*}{W}_\mathrm{stir} + \overset{*}{m}_\mathrm{b}(t) \cdot h_\mathrm{b} + \overset{*}{m}_\mathrm{p}(t) \cdot h_\mathrm{p} + \overset{*}{m}_\mathrm{pep} \cdot h_\mathrm{pep}.$$

   With the previously computed quantities we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\mathrm{r}(t) \cdot c_\mathrm{r} \cdot \vartheta_\mathrm{r}(t)\right) &= \alpha_\mathrm{r} \cdot A_\mathrm{r} \cdot (\vartheta_\mathrm{air} - \vartheta_\mathrm{r}(t)) \\ &\quad - \alpha_\mathrm{b} \cdot A_\mathrm{b}(t) \cdot (\vartheta_\mathrm{r}(t) - \vartheta_\mathrm{b}) - \alpha_\mathrm{p} \cdot A_\mathrm{p}(t) \cdot (\vartheta_\mathrm{r}(t) - \vartheta_\mathrm{p}) \\ &\quad + \overset{*}{W}_\mathrm{stir} \\ &\quad + \overset{*}{m}_\mathrm{b}(t) \cdot (c_\mathrm{b} \cdot \vartheta_\mathrm{m,b} + L_\mathrm{b}) + \overset{*}{m}_\mathrm{p}(t) \cdot (c_\mathrm{p} \cdot \vartheta_\mathrm{m,p} + L_\mathrm{p}) \\ &\quad + \overset{*}{m}_\mathrm{pep}(t) \cdot c_\mathrm{pep} \cdot \vartheta_\mathrm{pep}.\end{aligned}$$

   Remark. Notice that adding the peperonata to the hot rice mixture affects its internal energy and hence its quality. Through experimental validation over the last few centuries, the Risotto community has reached the conclusion that such ingredients should stay away from the beloved traditional recipe.