Exercise 3 - Lagrange Formalism

3.1 The Lagrange Formalism

The Lagrange formalism is a powerful tool that allows to derive the equations of motion of a mechanical system. This method offers several advantages over the reservoir based approach/Newton's laws if

- the system features multiple degrees of freedom/bodies;
- constraint forces are to be eliminated;
- no knowledge about internal forces is required;
- direct inclusion of non-holonomic constraints in the EoM is sought.

3.1.1 Generalized Coordinates

Generalized coordinates are signed displacements/angles that can be used to describe the configuration of a system. Conventionally, they are stored in a vector $\mathbf{q} \in \mathbb{R}^n$, arbitrarily ordered.

3.1.2 Holonomic and Non-holonomic Constraints

Constraints establish a mathematical relation between the generalized coordinates and their time derivatives. In general, non-holonomic constraints can be written as

$$f(\mathbf{q}, \dot{\mathbf{q}}, t) = 0. \tag{3.1}$$

A non-holonomic constraint can be thought as a restriction of the trajectory that the system takes to reach a certain configuration. When no dependency on $\dot{\mathbf{q}}$ is present, the constraint is called holonomic and reads

$$f(\mathbf{q}, t) = 0. \tag{3.2}$$

A holonomic constraint can be interpreted as a restriction of the reachable configurations that the system can take.

Example 1. For a pendulum of length $2L$ one may use $\mathbf{q} = \begin{bmatrix}\theta\end{bmatrix}$ or $\mathbf{q} = \begin{bmatrix}x & y\end{bmatrix}^\top$ subject to the constraint $f(x, y) = x^2 + y^2 - L^2 = 0$, where x and y are the coordinates of the center of mass of the pendulum. The latter equation is a holonomic constraint for the pendulum.

Degrees of Freedom

In general, it holds that

$$p = n - r,$$

where p is the number of degrees of freedom of a system, n the number of generalized coordinates used to describe the system and r the number of holonomic constraints acting on the system. Moreover, if a non-holonomic constraint can be integrated (is integrable), then it will become holonomic and eliminate degrees of freedom as well.

This material is based on the HS17 teaching assistance taught by Nicolas Lanzetti and Gioele Zardini.
This document can be downloaded at `https://n.ethz.ch/~lnicolas`

Example 2. Let's assume to have a constraint acting on a system described by n generalized coordinates q_i. If the constraint is holonomic, then it reads

$$f(q_1, \cdots, q_n, t) = 0,$$

which can be rewritten as

$$q_1 = \tilde{f}(q_2, \cdots, q_n, t),$$

which implies that the number of coordinates required is reduced by 1.

Minimal Coordinates

A set of generalized coordinates is called minimal if it contains exactly p coordinates.

Integrability

Every holonomic constraint f satisfies

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \sum_{i=1}^{n} \frac{\partial f(\mathbf{q}, t)}{\partial q_i} \dot{q}_i + \frac{\partial f(\mathbf{q}, t)}{\partial t} \\ &= \sum_{i=1}^{n} a_i(\mathbf{q}, t) \dot{q}_i + b(\mathbf{q}, t). \end{aligned} \tag{3.3}$$

Consider now the following example.

Example 3. Consider a wheel moving on a straight line without slipping. Here, we have $\mathbf{q} = \begin{bmatrix} x & \phi \end{bmatrix}^\top$. The non-slipping condition can be expressed by the constraint

$$\dot{x} = R\dot{\phi}.$$

This constraint might look non-holonomic, as it involves $\dot{x}$ and $\dot{\phi}$. However, we may integrate w.r.t. time both sides and rewrite the constraint as

$$x - x_0 = R(\phi - \phi_0).$$

From this last equation it is clear that the constraint is holonomic.

3.1.3 Conservative Systems

A system is called conservative if all forces $\mathbf{F}$ acting on it either do no work or can be expressed as the gradient of some scalar function U, called potential. That is,

$$\mathbf{F} = -\nabla U.$$

If one or more forces do not satisfy the above conditions, the system is not conservative.

3.1.4 The Lagrange Method

Given a mechanical system, the method reads as follows:

(I) Identify the vector of generalized coordinates $\mathbf{q}$.

(II) Is the system holonomic or non-holonomic? Are non-holonomic constraints integrable?

(III) Define the total kinetic and the potential energy of the system. Recall that the kinetic energy expressed with respect to the center of mass of the system reads

$$T = \underbrace{\frac{1}{2}m\dot{\mathbf{r}}(t)^2}_{\text{translation}} + \underbrace{\frac{1}{2}\Theta\omega(t)^2}_{\text{rotation}}$$

and that the potential energy for a mechanical system generally reads

$$U = mgh + \underbrace{\frac{1}{2}k_{\text{lin}}\Delta x^2}_{\text{linear spring}} + \underbrace{\frac{1}{2}k_{\text{rot}}\Delta\varphi^2}_{\text{torsional spring}} \quad . \tag{3.4}$$

(IV) Define the Lagrange function or Lagrangian

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q}),$$

where T is the total kinetic energy of the system and U is the total potential energy of the system.

(V) If there are nonconservative forces and/or torques acting on the system, these should be taken into account. In order to do this, we want to compute the generalized forces Q_k[1].

(VI) Write the Lagrange formalism for each generalized coordinate q_k. For holonomic systems this reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k, \qquad k = 1, \ldots, n, \tag{3.5}$$

where Q_k are the generalized forces of the system. These n equations are the equations of motion of system. They can then be written as

$$\mathbf{M}(\mathbf{q}, t) \cdot \ddot{\mathbf{q}} = \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{Q}, t), \tag{3.6}$$

where $\mathbf{M}$ is the so-called **mass matrix**.

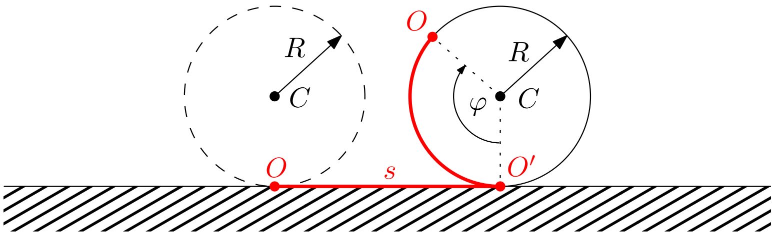
Remark. It is possible to use the Lagrange formalism also in case the system is subjected to non-holonomic constraints. Then, the Lagrange equations contain an additional term capturing this property. The treatment of the non-holonomic case is out of the scope of the System Modeling lecture and, as such, will not be part of the final exam.

[1] The computation of generalized forces is beyond of the scope of the System Modeling lecture. However, for the sake of completeness, we provide an efficient method in the Appendix of this document.

3.2 Tips

Exercise 1: For (e): be careful, S_1 is moving.

Exercise 2: No slipping condition:



3.3 Example

Due to the large demand, your SpaghETH shall increase the production. You decide to automate the mixing of pasta in water by carefully optimizing its operation. To do that, you start by formulating a model of the system. A sketch is shown in Figure 1.

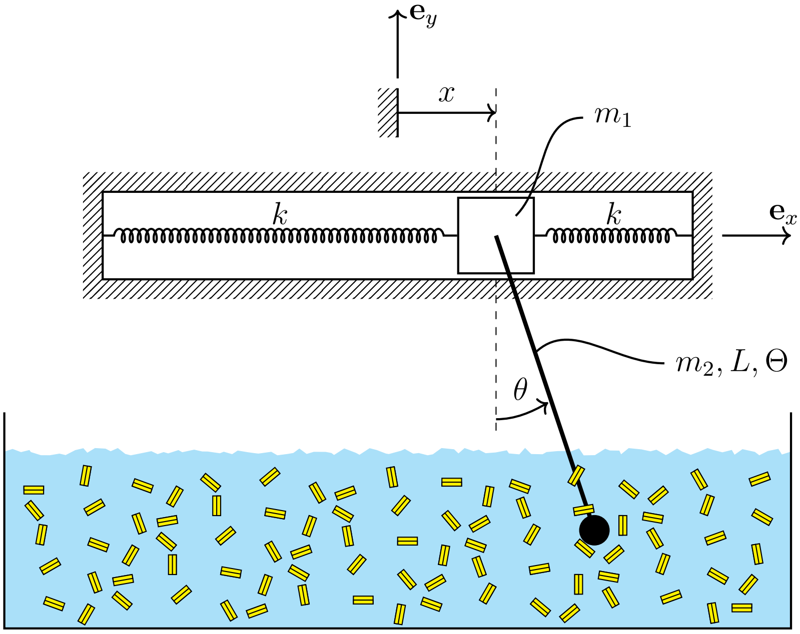


Figure 1: Sketch of the system.

The ladle is modeled as a bar of mass m_2, length L, and moment of inertia (w.r.t. its center of mass) $\Theta = \frac{1}{12}m_2L^2$. The ladle is attached to a point mass m_1. In order to deal with possible vibrations, the mass is attached to two springs with spring constant k. Assume the springs are unstretched at $x = 0$. Gravitational effects are to be considered; friction, aerodynamic losses as well as water resistance are to be neglected.

1. How many degrees of freedom does the system have? List at least two possible choices of generalized coordinates.
2. Is the system holonomic?
3. Is the system conservative?
4. Determine the kinetic energy of the system.
5. Determine the potential energy of the system.
6. Determine the equations of motion of the system.
7. Describe *qualitatively* how your answer would change if
   (a) an external force F acts horizontally on the mass m_1.
   (b) the system is brought in an electric field.

Solution.

1. The system has two degrees of freedom. Possible choices of generalized coordinates are (x, θ) and (x, ϕ), where ϕ denotes the angle between the bar and the $\mathbf{e}_x$-direction. Note that (x, x_{bar}) and (x, y_{bar}) are not valid minimal coordinate choices as they do not describe the position of the mass uniquely.

2. No constraints depend on $\dot{\mathbf{q}}$. Thus, the system is holonomic.

3. All forces acting on the system are potential (or do no work), hence the system is conservative.

4. Let $\mathbf{r}_1(t)$ be the position vector of the mass m_1 and $\mathbf{r}_2(t)$ the position of the center of mass of the bar. In order to compute the kinetic energy, we first compute the velocities of the mass m and of the bar. In an inertial frame we get

$$\mathbf{r}_1(t) = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{r}_2(t) = \begin{bmatrix} x + \frac{L}{2}\sin(\theta) \\ -\frac{L}{2}\cos(\theta) \\ 0 \end{bmatrix}.$$

The velocities are then

$$\dot{\mathbf{r}}_1(t) = \begin{bmatrix} \dot{x} \\ 0 \\ 0 \end{bmatrix}, \qquad \dot{\mathbf{r}}_2(t) = \begin{bmatrix} \dot{x} + \frac{L}{2}\dot{\theta}\cos(\theta) \\ \frac{L}{2}\dot{\theta}\sin(\theta) \\ 0 \end{bmatrix}.$$

The angular velocity of the bar is given by

$$\boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}.$$

Thus, the kinetic energy of the system is

$$\begin{aligned} T &= \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 + \frac{1}{2}\Theta|\boldsymbol{\omega}|^2 \\ &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\left(\dot{x} + \frac{L}{2}\dot{\theta}\cos(\theta)\right)^2 + \frac{L^2}{4}\dot{\theta}^2\sin^2(\theta)\right) + \frac{1}{2}\frac{1}{12}m_2L^2\dot{\theta}^2 \\ &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + \frac{L^2}{4}\dot{\theta}^2 + \dot{x}L\dot{\theta}\cos(\theta)\right) + \frac{1}{2}\frac{1}{12}m_2L^2\dot{\theta}^2. \end{aligned}$$

Remark. Alternatively, the velocity of the center of mass of the bar can be computed with

$$\dot{\mathbf{r}}_2 = \dot{\mathbf{r}}_1 + \boldsymbol{\omega} \times \mathbf{r}_{12} = \begin{bmatrix} \dot{x} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \times \begin{bmatrix} \frac{L}{2}\sin(\theta) \\ -\frac{L}{2}\cos(\theta) \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{x} + \frac{L}{2}\dot{\theta}\cos(\theta) \\ \frac{L}{2}\dot{\theta}\sin(\theta) \\ 0 \end{bmatrix}.$$

5. The potential energy of the system is

$$U = \frac{1}{2}kx^2 + \frac{1}{2}k(-x)^2 - m_2g\frac{L}{2}\cos(\theta) = kx^2 - m_2g\frac{L}{2}\cos(\theta).$$

6. The Lagrangian is defined as

$$\begin{aligned}\mathcal{L} &= T - V \\ &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + \frac{L^2}{4}\dot{\theta}^2 + \dot{x}L\dot{\theta}\cos(\theta)\right) + \frac{1}{2}\frac{1}{12}m_2L^2\dot{\theta}^2 - kx^2 + m_2g\frac{L}{2}\cos(\theta).\end{aligned}$$

The derivatives are

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{x}} &= m_1\dot{x} + m_2\dot{x} + \frac{1}{2}m_2L\dot{\theta}\cos(\theta), \\ \frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= m_2\frac{L^2}{4}\dot{\theta} + \frac{1}{2}m_2L\dot{x}\cos(\theta) + \frac{1}{12}m_2L^2\dot{\theta}, \\ \frac{\partial\mathcal{L}}{\partial x} &= -2kx, \\ \frac{\partial\mathcal{L}}{\partial\theta} &= -\frac{1}{2}m_2\dot{x}L\dot{\theta}\sin(\theta) - m_2g\frac{L}{2}\sin(\theta).\end{aligned}$$

The equation of motion for the generalized coordinate x is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(m_1\dot{x} + m_2\dot{x} + \frac{1}{2}m_2L\dot{\theta}\cos(\theta)\right) - (-2kx) = 0 \\ \Rightarrow\quad &(m_1+m_2)\ddot{x} + \frac{1}{2}m_2L\left(\ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta)\right) + 2kx = 0.\end{aligned}$$

The equation of motion for the generalized coordinate θ is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(m_2\frac{L^2}{4}\dot{\theta} + \frac{1}{2}m_2L\dot{x}\cos(\theta) + \frac{1}{12}m_2L^2\dot{\theta}\right) - \left(-\frac{1}{2}m_2\dot{x}L\dot{\theta}\sin(\theta) - m_2g\frac{L}{2}\sin(\theta)\right) = 0 \\ \Rightarrow\quad &m_2\left(\frac{L^2}{4} + \frac{L^2}{12}\right)\ddot{\theta} + \frac{1}{2}m_2L\left(\ddot{x}\cos(\theta) - \dot{x}\dot{\theta}\sin(\theta)\right) + \frac{1}{2}m_2\dot{x}L\dot{\theta}\sin(\theta) + m_2g\frac{L}{2}\sin(\theta) = 0.\end{aligned}$$

7. (a) The system would not be conservative anymore. The force F has to be considered as generalized force Q_1 acting on the generalized coordinate x and as generalized force Q_2 acting on the generalized coordinate θ.

   (b) The system is still conservative. The electric potential can be included in U.

A Computation of Generalized Forces with Jacobians

Suppose to have a force $\mathbf{F}$ acting on point A of the system. The velocity of point A can always be written as

$$\mathbf{v}_A = \mathbf{J}_A \dot{\mathbf{q}} + \boldsymbol{\nu}_A,$$

where $\mathbf{J}_A$ is the translational Jacobian matrix of point A, $\mathbf{q}$ is the generalized-coordinates vector, and $\boldsymbol{\nu}_A$ is an offset term. The resulting general force can be written as

$$\mathbf{Q}_A = \mathbf{J}_A^\top \mathbf{F}.$$

Suppose to have a torque $\mathbf{M}$ acting on the body. The angular velocity of the system $\boldsymbol{\omega}$ can always be written as

$$\boldsymbol{\omega} = \mathbf{J}_R \dot{\mathbf{q}} + \boldsymbol{\nu}_E,$$

where $\mathbf{J}_R$ is the rotational Jacobian matrix and $\boldsymbol{\nu}_E$ is an offset term. The resulting general force can be written as

$$\mathbf{Q}_R = \mathbf{J}_R^\top \mathbf{M}.$$

Example (Adapted from Exam HS2016)

Consider the mechanical system depicted in Figure 2. Let the vector of (minimal) generalized coordinates be $\mathbf{q} = \begin{bmatrix}\alpha & \beta\end{bmatrix}^\top$. Calculate the generalized force vector $\mathbf{Q}_2$ associated with the force $\mathbf{F}_2$.

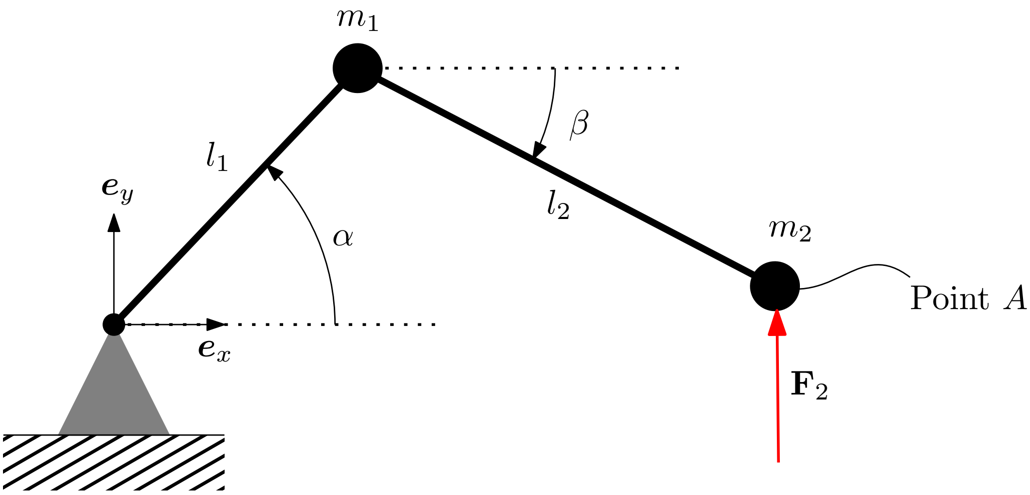


Figure 2: Sketch of the system.

Solution. The velocity of point A can be written as:

$$\mathbf{v}_A = \mathbf{J}_A \dot{\mathbf{q}} + \boldsymbol{\nu}_A.$$

Therefore, one way to obtain the Jacobian is to compute the velocity of point A as a function of the generalized velocities and read out the matrix. First, we compute the position vector as

$$\mathbf{r}_{OA} = \begin{bmatrix} l_1\cos(\alpha) + l_2\cos(\beta) \\ l_1\sin(\alpha) - l_2\sin(\beta) \\ 0 \end{bmatrix},$$

the velocity is then given by

$$\mathbf{v}_A = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_{OA}) = \begin{bmatrix} -l_1\dot{\alpha}\sin(\alpha) - l_2\dot{\beta}\sin(\beta) \\ l_1\dot{\alpha}\cos(\alpha) - l_2\dot{\beta}\cos(\beta) \\ 0 \end{bmatrix},$$

which can be rewritten as a matrix multiplication in the following way:

$$\mathbf{v}_A = \underbrace{\begin{bmatrix} -l_1\sin(\alpha) & -l_2\sin(\beta) \\ l_1\cos(\alpha) & -l_2\cos(\beta) \\ 0 & 0 \end{bmatrix}}_{\mathbf{J}_A} \cdot \underbrace{\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}}_{\dot{\mathbf{q}}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{\nu}_A}.$$

This formulation allows us to read out the required quantities directly.

The generalized force $\mathbf{Q}_2$ is then given by

$$\mathbf{Q}_2 = \mathbf{J}_A^\top \cdot \mathbf{F}_2 = \begin{bmatrix} -l_1\sin(\alpha) & l_1\cos(\alpha) & 0 \\ -l_2\sin(\beta) & -l_2\cos(\beta) & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ F_2 \\ 0 \end{bmatrix} = \begin{bmatrix} F_2 l_1\cos(\alpha) \\ -F_2 l_2\cos(\beta) \\ 0 \end{bmatrix}.$$

Remark. If we were to compute the generalized force corresponding to a torque, we would apply the same procedure with the angular velocity $\boldsymbol{\omega}$ of the body.