Exercise 2 - Mechanical Systems

2.1 Mechanical Systems

We start our analysis of systems by considering mechanical systems. In order to model such systems with the tools presented last week, we present a quick recap on how to compute energies and power. Of course, these tools represent only an alternative to the methods taught in courses such as Dynamics and Advanced Dynamics.

2.1.1 Kinetic Energy

The kinetic energy of a rigid body is denoted with T and can be expressed as the sum of the translational and the rotational kinetic energy. That is,

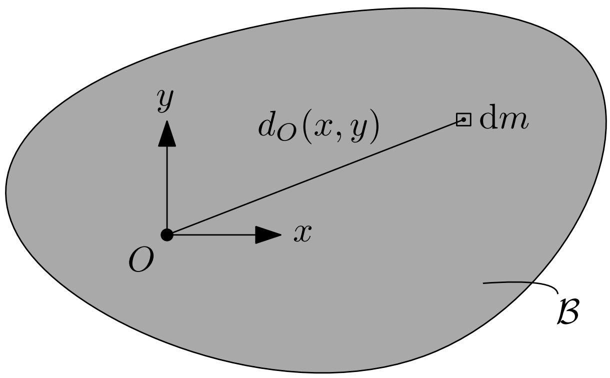
$$\begin{aligned} T(t) &= T_\mathrm{t}(t) + T_\mathrm{r}(t) \\ &= \frac{1}{2} m \dot{\mathbf{r}}(t)^2 + \frac{1}{2} \Theta \omega(t)^2, \end{aligned} \tag{2.1}$$

where $\mathbf{r}$ is the position vector of the center of mass, Θ is the moment of inertia with respect to the center of mass, and ω is the angular velocity of the body. Note that $\dot{\mathbf{r}}^2 = \dot{\mathbf{r}}^\top \dot{\mathbf{r}}$.

Remark. Equation (2.1) is a simplified version of the kinetic energy formula for 2D rigid bodies.

Remark. Recall that moment of inertia for a 2D rigid body $\mathcal{B}$ with respect to an axis through point O is defined as

$$\int_{\mathcal{B}} d_O(x, y)^2 \, \mathrm{d}m.$$



It follows by definition that the moment of inertia is an additive quantity. That is, given two masses with moments of inertia Θ_1 and Θ_2 w.r.t. the same point P, the total moment of inertia w.r.t P is simply given by $\Theta_\mathrm{tot} = \Theta_1 + \Theta_2$.

2.1.2 Potential Energy

The potential energy of a system is a sole function of $\mathbf{r}(t)$, i.e.,

$$U(t) = U(\mathbf{r}(t)). \tag{2.2}$$

Practical examples are the gravitational potential energy:

$$U_\mathrm{g} = mgh, \tag{2.3}$$

This material is based on the HS17 teaching assistance taught by Nicolas Lanzetti and Gioele Zardini.
This document can be downloaded at `https://n.ethz.ch/~lnicolas`

and the spring potential energy:

$$U_{\text{spring}} = \frac{1}{2}kx^2, \tag{2.4}$$

where g is the gravitational acceleration and k is the spring constant and x the displacement from the relaxed position of the spring.

2.2 Mechanical Systems: Reservoir-based Approach

We can directly apply the reservoir-based approach to mechanical systems. Here, reservoirs consist of:

- Kinetic energies, whose level variables are typically velocities and/or
- Potential energies, whose level variables are typically positions or angles.

Flows are then given in terms of powers. We distinguish between the power of a force $\mathbf{F}$:

$$P_F = \mathbf{F}^\top \mathbf{v}, \tag{2.5}$$

and the power of a torque $\mathbf{T}$:

$$P_T = \mathbf{T}^\top \boldsymbol{\omega}, \tag{2.6}$$

where $\mathbf{v}$ is the velocity of the point of application of the force and $\boldsymbol{\omega}$ the angular velocity of the body.

Remark. Generally, one uses (2.6) for free torques. Be careful not to account for both the power of a force and the power of the torque such force produces.

Then, for each reservoir we may proceed as usual with

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = P_+(t) - P_-(t). \tag{2.7}$$

Typical examples of forces acting on mechanical systems are:

- **Gravitational force**, given by
$$F_{\mathrm{g}} = mg; \tag{2.8}$$
- **spring force**, given by
$$F_{\text{spring}} = kx; \tag{2.9}$$
- **rolling friction**, approximated as
$$F_{\mathrm{r}} = c_{\mathrm{r}}mg, \tag{2.10}$$
where c_{r} is the rolling friction coefficient. Note that this force is dissipative in nature and acts in the opposite direction of the motion of the rolling object.
- **Aerodynamic drag force**, expressed by
$$F_{\mathrm{a}} = \frac{1}{2}\rho c_{\mathrm{w}} A v^2, \tag{2.11}$$
where ρ is the surrounding fluid density, c_{w} the drag coefficient, A the projected surface of the moving object (also known as *apparent area*) and v the relative velocity of the object w.r.t. the surrounding fluid.

Remark. For systems with only one degree of freedom it might be easier to directly use

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{tot}}(t) = \sum_i P_i(t),$$

where $E_{\text{tot}} = E_{\text{kin,tot}} + U_{\text{tot}}$ is the total energy of the system.

Example 1. Consider the mechanical oscillator depicted in Figure 1.

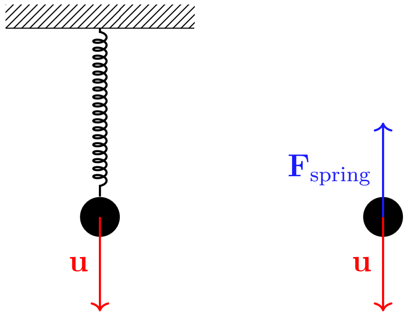


Figure 1: Mechanical oscillator.

The reservoirs are:

- the kinetic energy with level variable v and
- the potential energy of the spring with level variable x.

The conservation laws read, respectively

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right) = u \cdot v - kx \cdot v$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}kx^2\right) = kx \cdot v.$$

Then, some basic algebraic manipulations lead to

$$\begin{aligned} m\dot{v} &= u - kx \\ \dot{x} &= v, \end{aligned}$$

or, equivalently, to

$$m\ddot{x} = u - kx.$$

2.3 Tips

Exercise 1: No tips. ☺

Exercise 2: Compute the kinetic energies in horizontal and vertical directions separately.

2.4 Example

Since your company SpaghETH is going well, you decide to improve the service offered by purchasing a crane from your colleagues at CranETH. The idea is to efficiently distribute the pots with hot water to the truck drivers. A sketch of the crane is shown in Figure 2.

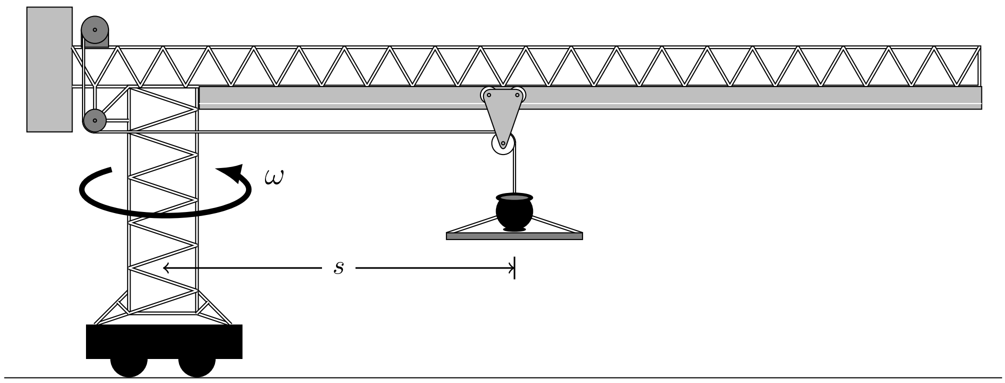


Figure 2: Sketch of the system.

The crane platform has negliglible mass. The crane itself has mass m_c and moment of inertia with respect to the vertical axis Θ. The crane has front surface A, the density of air is known and is given by ρ, the aerodynamic coefficient is c_w and the rolling friction coefficient c_r. Additionally, a frictional torque, expressed as $T_\mathrm{fric} = \beta\omega$, counteracts the crane's rotation.
Experiments have shown that the aerodynamic drag coeffiecient is a function of the rotational velocity of the crane. The crane is carrying a pot of mass m_p which is attached at an adjustable distance s from the vertical axis. You may treat the pot as a point mass. Furthermore, assume that the center of mass of the system does not change as the mass m_p moves, i.e. it always lies on the vertical axis of the crane. The propulsive force acting horizontally on the crane and the propulsive torque acting on the crane vertical axes are given by

$$
\begin{aligned}
F_\mathrm{p}(\phi_1) &= F_\mathrm{max} \cdot (1 - \exp(-c_1\phi_1)) \\
T_\mathrm{p}(\phi_2) &= T_\mathrm{max} \cdot (1 - \exp(-c_2\phi_2)),
\end{aligned}
$$

where $\phi_1(t)$ and $\phi_2(t)$ are the normalized actuators positions. The constants $P_\mathrm{max}, T_\mathrm{max}, c_1$, and c_2 are known. Finally, assume that the rope force always balances the weight of the pot.

1. Determine the inputs and the outputs of the system.
2. List the reservoir(s) and the corresponding level variable(s).
3. Draw a causality diagram of the system.
4. Formulate the differential/algebraic equations needed to describe the system.
5. Is the system linear or nonlinear? Explain.

Solution.

1. The inputs are the actuator values ϕ_1 and ϕ_2 as well as the distance s of the pot from the central axis. The outputs are the translational and rotational velocities of the system.

2. The system has two reservoirs:

   - the kinetic translational energy of the system E_{tr}, whose level variable is the velocity v of the system;
   - the kinetic rotational energy of the system E_{rot}, whose level variable is the rotational velocity ω of the system.

   As the rope force balances the weight of the pot the gravitational potential energy of the pot is not a reservoir.

3. The causality diagram is shown in Figure 3.

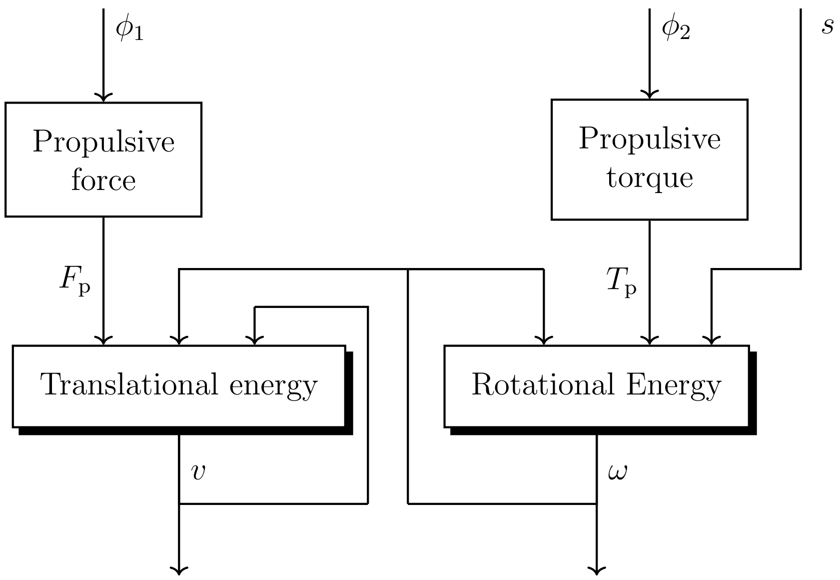


Figure 3: Causality diagram of the system.

4. The differential equation for the translational energy of the truck reads

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{tr}} = P_+ - P_-,$$

which reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{tr}} = F_{\text{p}}v - \frac{1}{2}\rho c_{\text{w}}(\omega)Av^3 - c_{\text{r}}(m_{\text{c}} + m_{\text{p}})gv.$$

This leads to the differential equation

$$(m_{\text{c}} + m_{\text{p}})v\dot{v} = F_{\text{p}}v - \frac{1}{2}\rho c_{\text{w}}(\omega)Av^3 - c_{\text{r}}(m_{\text{c}} + m_{\text{p}})gv$$

which simplifies to

$$\dot{v} = \frac{1}{m_\mathrm{c} + m_\mathrm{p}} \cdot \left(F_\mathrm{p} - \frac{1}{2}\rho c_\mathrm{w}(\omega) A v^2 - c_\mathrm{r}(m_\mathrm{c} + m_\mathrm{p}) g \right).$$

The differential equation for the rotational energy of the crane reads

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\mathrm{rot} = P_+ - P_-,$$

which reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\mathrm{rot} = T_\mathrm{p}\omega - \beta\omega^2.$$

The rotational energy of the system is

$$E_\mathrm{rot} = \frac{1}{2}(\Theta + m_\mathrm{p} s^2)\omega^2.$$

Hence, the differential equation reads

$$(\Theta + m_\mathrm{p} s^2)\omega\dot{\omega} + m_\mathrm{p}\omega^2 s\dot{s} = T_\mathrm{p}\omega - \beta\omega^2,$$

which simplifies to

$$\dot{\omega} = \frac{1}{(\Theta + m_\mathrm{p} s^2)} \cdot \left(T_\mathrm{p} - \beta\omega - m_\mathrm{p}\omega s\dot{s} \right).$$

5. The system is nonlinear, in fact the dynamics of both the translational and rotational velocities as well as the input dependencies are nonlinear.