Exercise 8 - Recap

1 Angular Momentum

For the angular momentum we use the classic definition:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \quad \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2.$$

We have seen that $\hat{L}_x, \hat{L}_y$, and $\hat{L}_z$ do not commute, but they all commute with $\hat{L}^2$. The eigenfunctions are the spherical harmonics and the eigenvalues are function of l:

$$\hat{L}^2 Y_l^{m_l} = \hbar^2 l(l+1) Y_l^{m_l},$$
$$\hat{L}_z Y_l^{m_l} = \hbar m_l Y_l^{m_l},$$

where $l = 0, 1, 2, \ldots$ and $m_l = -l, -l+1, \ldots, l-1, l$.

2 Spin

Due the intrisic spin of the particle, it acts *as if* it has an inherent rotation about z-axis. Similarily to the angular momentum is eigenvalue problem is:

$$\hat{S}^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle,$$
$$\hat{S}_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle.$$

for $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $m_s = -s, -s+1, \ldots, s-1, s$.

Remark. For an electron: it can occupy different orbitals, so l can vary ($s, d, p, \ldots$ orbitals). However, each quantum mechanical particle has a fixed spin s. For an electron $s = \frac{1}{2}$, for a photon $s = 1$.

Therefore, for an electron (or any $\frac{1}{2}$-spin particle) there are two possible eigenstates:

$$|s, m_s\rangle \to \begin{cases}]\frac{1}{2}, +\frac{1}{2}\rangle & \text{“spin up”}, \\ |\frac{1}{2}, -\frac{1}{2}\rangle & \text{“spin down”}, \end{cases}$$

where up/down refer to the projection of the spin along the z-axis. We can choose these eigenstates as out basis vectors, that is

$$|\tfrac{1}{2}, +\tfrac{1}{2}\rangle \to \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then, the general sping state can be written as a linear combination:

$$|\chi\rangle = a|\tfrac{1}{2}, +\tfrac{1}{2}\rangle + b|\tfrac{1}{2}, -\tfrac{1}{2}\rangle,$$
$$|\chi\rangle = a\begin{bmatrix} 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$|\chi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

The operators in this basis are given by

$$\hat{S}^2 = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \hat{S}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \hat{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Example. Let's work out $\hat{S}^2$. We know it is a 2×2 matrix, that is

$$\hat{S}^2 = \begin{bmatrix} c & d \\ e & f \end{bmatrix}.$$

We use the eigevalue equations:

$$\begin{aligned} \hat{S}^2|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle &= \hbar^2 s(s+1)|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle \\ &= \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right)|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle \\ &= \frac{3}{4}\hbar^2|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle. \end{aligned}$$

In matrix notation:

$$\begin{bmatrix} c & d \\ e & f \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} c \\ e \end{bmatrix} = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus, $c = \frac{3}{4}\hbar^2$ and $e = 0$. Similarily

$$\begin{bmatrix} c & d \\ e & f \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{4}\hbar^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} d \\ f \end{bmatrix} = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which leads to $d = 0$ and $f = \frac{3}{4}\hbar^2$. Hence,

$$\hat{S}^2 = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example. Consider a particle with spin 2. The possible values of m_s are $\{-2, -1, 0, 1, 2\}$. Therefore, there are 5 eigenstates, given by $|2, \pm 2\rangle$, $|2, \pm 1\rangle$, and $|2, 0\rangle$. In vector form we would have 5-dimensional vectors/matrices.