Exercise 4 - Recap

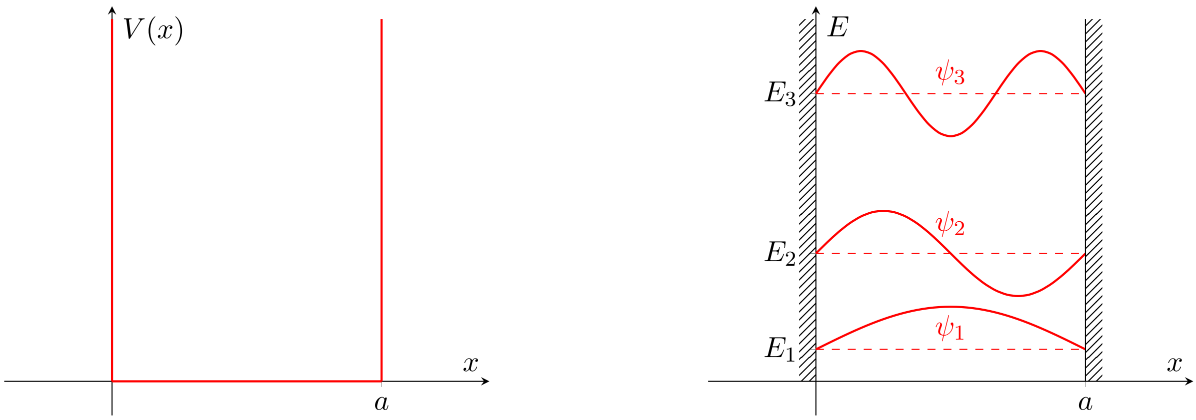
1 Infinite Square Well

Consider a particle in a infinite square well:

$$V(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a, \\ \infty & \text{else,} \end{cases} \tag{1.1}$$

i.e. the particle can only be between 0 and a. The solutions to the TISE are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right), \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}. \tag{1.2}$$



(a) Potential energy function. (b) Wave functions.

Figure 1: Infinite Square Well.

2 Quantum Harmonic Oscillator

Consider a particle in the potential $V(x) = \frac{1}{2}m\omega^2x^2$. The TISE yiels

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi. \tag{2.1}$$

To solve for ψ_n we introduce

$$\begin{aligned} \hat{a}_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x}) = \text{raising operator}, \\ \hat{a}_- &= \frac{1}{\sqrt{2\hbar m\omega}}(+i\hat{p} + m\omega\hat{x}) = \text{lowering operator}. \end{aligned} \tag{2.2}$$

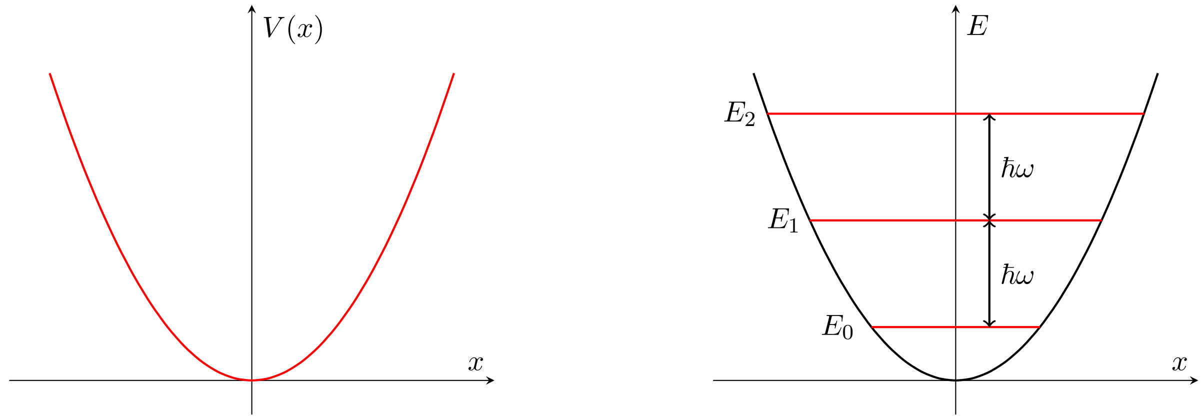
Why? To make the TISE easier to solve. In particular, if ψ solve the TISE with energy E, then

$$\begin{aligned} \hat{H}\hat{a}_+\psi &= (E + \hbar\omega)\hat{a}_+\psi, \\ \hat{H}\hat{a}_-\psi &= (E - \hbar\omega)\hat{a}_-\psi, \end{aligned}$$

i.e. $\hat{a}_{\pm}\psi$ solves the TISE with energy $E \pm \hbar\omega$. From that:

$$\hat{a}_+\psi_n = \sqrt{n+1}\psi_{n+1}, \qquad \psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right), \qquad E_n = \left(n + \frac{1}{2}\right)\hbar\omega,$$
$$\hat{a}_-\psi_n = \sqrt{n}\psi_{n-1}, \qquad \psi_n = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n\psi_0,$$

where ψ_0 comes from $\hat{a}_-\psi = 0$.



(a) Potential energy function.

(b) Wave functions.

Figure 2: Quantum Harmonic Oscillator.