Exercise 1 - The Momentum Operator

The momentum operator is

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}.$$

What does that mean?
The expectation value of the position of the particle is

$$\langle x\rangle = \int_{-\infty}^{+\infty} x|\Psi|^2\,\mathrm{d}x.$$

The expectation value of the momentum is therefore

$$\begin{aligned}
m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} &= m\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} x|\Psi|^2\,\mathrm{d}x \\
&= m\int_{-\infty}^{+\infty} x\frac{\partial}{\partial t}\Psi^*\Psi\,\mathrm{d}x \\
&= m\int_{-\infty}^{+\infty} x\left(\frac{\partial\Psi^*}{\partial t}\Psi + \frac{\partial\Psi}{\partial t}\Psi^*\right)\mathrm{d}x \\
&= m\int_{-\infty}^{+\infty} x\left(-\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + V\Psi^*\right)\Psi + \frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right)\Psi^*\right)\mathrm{d}x \\
&= m\int_{-\infty}^{+\infty} \frac{x}{i\hbar}\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi^*}{\partial x^2}\Psi - \frac{\partial^2\Psi}{\partial x^2}\Psi^*\right)\mathrm{d}x \\
&= m\int_{-\infty}^{+\infty} \left(-\frac{i\hbar}{2m}\right)x\frac{\partial}{\partial x}\left(\frac{\partial\Psi^*}{\partial x}\Psi - \frac{\partial\Psi}{\partial x}\Psi^*\right)\mathrm{d}x \\
&= -\frac{i\hbar}{2}\left(x\left(\frac{\partial\Psi^*}{\partial x}\Psi - \frac{\partial\Psi}{\partial x}\Psi^*\right)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\left(\frac{\partial\Psi^*}{\partial x}\Psi - \frac{\partial\Psi}{\partial x}\Psi^*\right)\mathrm{d}x\right) \\
&= \frac{i\hbar}{2}\left(\int_{-\infty}^{+\infty}\frac{\partial\Psi^*}{\partial x}\Psi\,\mathrm{d}x - \int_{-\infty}^{+\infty}\frac{\partial\Psi}{\partial x}\Psi^*\,\mathrm{d}x\right) \\
&= \frac{i\hbar}{2}\left(\Psi^*\Psi\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\frac{\partial\Psi}{\partial x}\Psi^*\,\mathrm{d}x - \int_{-\infty}^{+\infty}\frac{\partial\Psi}{\partial x}\Psi^*\,\mathrm{d}x\right) \\
&= -i\hbar\int_{-\infty}^{+\infty}\frac{\partial\Psi}{\partial x}\Psi^*\,\mathrm{d}x \\
&= \int_{-\infty}^{+\infty}\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi\,\mathrm{d}x.
\end{aligned}$$

Thus, it holds $\hat{p} = -i\hbar\frac{\partial}{\partial x}$.