A posteriori estimator for the accuracy of the shallow shelf approximation

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Received: date / Accepted: date

Abstract In this paper, we prove a “mechanical” estimator that can evaluate the solution discrepancy between two embedded ice flow models: the first order approximation and the shallow shelf approximation. To do so, we apply residual techniques as those employed to derive a posteriori estimators of the numerical error. Numerical experiments validate the ability of our estimator to predict the accuracy of the shallow shelf approximation model.

Keywords A posteriori estimator · non-Newtonian fluid · p-Laplace equation · shallow ice models

1 Introduction

In most of glacier and ice sheet models, ice is described as a viscous non-Newtonian fluid such that its motion is governed by 3D nonlinear Stokes equations [11]. However, it is common practice to simplify ice flow equations by neglecting the highest-order terms in the aspect ratio of the ice domain since this ratio is usually small. As a result, the First-Order Approximation [2] (FOA) consists of a 3D nonlinear elliptic equation [5,17] for the horizontal velocity, the pressure and the vertical component of the velocity being eliminated after simplifications. When the ice flow is dominated by sliding as it occurs over ice shelves, it is common to further assume no vertical variation of the velocity. The resulting model is called the Shallow Shelf Approximation [13,12] (SSA) and consists of a cheaper 2D nonlinear elliptic equation [16] for the horizontal velocity.

In practice, using one or another model is a matter of empirical modelling choice, which considers the type of ice flow and the computational resource available. Through modelling exercises like ISMIP-HOM [14], the validity of the SSA
against the FOA could be assessed for different types of ice flow. However, there exist no mathematical result that can estimate the accuracy of the SSA against the FOA with respect to model set-up data. To fill this gap, we derive in this paper an \textit{a posteriori} error estimate that evaluates the solution discrepancy between the FOA and the SSA models. Our approach involves residual techniques similar to the ones involved to derive \textit{a posteriori} estimators of the numerical error [15]. Here, the most challenging task is to obtain an estimate that does not depend on the FOA solution, which is the most expensive solution. For this reason, we call “\textit{a posteriori}” such an estimator.

The outlines of this paper are: In section 2, we write out the FOA and SSA ice flow models. Then, we formulate the two models as variational and minimisation problems in section 3, and derive a “mechanical estimator” in section 4. Finally, in section 5, we test the capability of the estimator to evaluate the accuracy of the SSA in a simple numerical experiment.

2 Model

Let \( V \) be a three-dimensional domain of ice, which is defined by

\[ V = \{(x, y, z), \ s(x, y) \leq z \leq \bar{s}(x, y)\}, \]

where \( x = (x, y) \) denote the horizontal coordinates, \( z \) denotes the vertical coordinate, \( s(x) \), \( \bar{s}(x) \) and \( h(x) := \bar{s}(x) - s(x) \) are the elevation of the bedrock, the elevation of the upper ice surface, and the ice thickness, see Fig. 1. The boundary of \( V \) is divided into the ice-air interface called

\[ \Gamma_s = \{(x, y, z), \ z = \bar{s}(x, y)\}, \]

and the ice-bedrock interface

\[ \Gamma_0 \cup \Gamma_m = \{(x, y, z), \ z = s(x, y)\}. \]

On this latter, ice might be stuck to the ground or slide, see Fig. 1. For this reason, we distinguish two cases and we call \( \Gamma_0 \) the non-sliding and \( \Gamma_m \) the sliding parts of the ice-bedrock interface. Finally, we have \( \partial V = \Gamma_s \cup \Gamma_0 \cup \Gamma_m \).

In this paper we mostly follow the notations introduced in [17]. In addition, \( \nabla_x \) denotes the gradient with respect to horizontal coordinates \( x \) while \( \nabla \cdot \) denote the divergence in all coordinates \( (x, y, z) \). Furthermore, \( \partial_k \) denotes the partial derivative with respect to \( k \in \{x, y, z\} \).

The First Order Approximation (FOA) model (isothermal version) describes the horizontal velocity field of ice \( u = (u_x, u_y) \) by the following non-linear elliptic equation [5,15,17]:

\[ -B \nabla \cdot \left( \|D(u)\|^{p-2}_a \left[D(u) + \text{tr}(D(u))I\right] \right) = -\rho g \nabla_x \bar{s} \quad \text{in} \quad V, \]

where \( B > 0, \ p \in (1, 2] \) are two constant parameters,

\[ D(u) := \begin{pmatrix} \frac{1}{2} (\partial_x u_x + \partial_y u_y) & \frac{1}{2} (\partial_y u_x + \partial_z u_y) & \frac{1}{2} (\partial_z u_x) \\ 0 & \partial_y u_y & \frac{1}{2} (\partial_z u_y) \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \]

\( \partial_k \) denote the partial derivatives with respect to \( k \) in \( (x, y, z) \).
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Fig. 1 Cross-section of a glacier, with notations.

\[ \text{tr}(D(u)) := \partial_y u_y + \partial_x u_x, \] the inner product \((\cdot, \cdot)_s\) and its associated norm \(|\cdot|_s\) on \(\mathbb{R}^{2\times 3}\), the space of real symmetric 2-by-3 matrices, are defined by:

\[
(A, B)_s = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{3} A_{ij} B_{ij} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} A_{ii} B_{jj}, \quad |A|_s = \sqrt{(A, A)_s}, \tag{6}
\]

g is the gravitational constant and \(\rho\) is the ice density. The boundary conditions that supplement (4) are the following. No force applies on the ice-air interface,

\[
B|D(u)|^{p-2}_s [D(u) + \text{tr}(D(u)) I] \cdot n = 0, \quad \text{on } \Gamma_s, \tag{7}
\]

where \(n\) is an outer normal vector along \(\Gamma_s\). Along the lower surface interface, the no-slip condition is

\[
u = 0, \quad \text{on } \Gamma_0, \tag{8}
\]

and the non-linear friction condition reads [8]:

\[
B|D(u)|^{p-2}_s [D(u) + \text{tr}(D(u)) I] \cdot n = -C|u|^{s-2}|u|, \quad \text{on } \Gamma_m, \tag{9}
\]

where \(s \in (1, p], C = C(x, y) > 0, \) and \(n\) is the outward normal unit vector to \(\Gamma_m\).

The Shallow Shelf Approximation [13,12] (SSA) further assumes no vertical variation of the velocity in the FOA, namely

\[
\partial_z u = 0. \tag{10}
\]

Thus, the SSA suits ice flow dominated by basal sliding as over floating ice shelves. The SSA is usually vertically integrated and written as a 2D nonlinear elliptic equation [16] for the horizontal velocity. In what follows, we call \(u_S\) the SSA solution.

Along the paper, we denote by \(c\) a generic positive constant, which neither depends on FOA and SSA solutions \(u\) and \(u_S\), nor on physical parameters.
3 Variational and minimisation formulations

We now reformulate the FOA and the SSA as variational and minimisation problems following \cite{15,17}. For that, we define the following functional space:

\[
\mathcal{X} := \{ \mathbf{u} = (u_x, u_y) \in [W^{1,p}(V)]^2, \quad \mathbf{u} = 0 \text{ on } \Gamma_0 \}. \tag{11}
\]

In addition, we introduce several norms: \(\|\cdot\|_{L^p}\) and \(\|\cdot\|_{W^{1,p}}\) denote the standard norms of space \(L^p\) and \(W^{1,p}\) functional spaces while \(\|\cdot\|_{W^{1,p}}\) denotes the semi-norm defined by \(\|v\|_{W^{1,p}} = \|\nabla v\|_{L^p}\). From now on, we assume \(\Gamma_0\) has a non-zero measure, so that Poincaré inequality \cite{3} implies that \(\|\cdot\|_{W^{1,p}}\) and \(\|\cdot\|_{W^{1,p}}\) are equivalent on the functional space \(\mathcal{X}\) while Korn’s inequality \cite{6} implies the equivalence of norms \(\|\cdot\|_{W^{1,p}}\) and the one defined by \(\|D(\cdot)\|_{L^p}\) on space \(\mathcal{X}\).

After multiplying (4) by a test function \(v \in \mathcal{X}\) and integrating over \(V\), using Gauß’s theorem and the boundary conditions (7), (8), (9), the variational problem of the FOA writes \cite{5,17}:

Find \(\mathbf{u} \in \mathcal{X}\) such that

\[
\int_V B[D(\mathbf{u})]^{p-2}(D(\mathbf{u}), D(v))_s dV + \int_{\Gamma_m} C|\mathbf{u}|^{p-2}(\mathbf{u} \cdot v)_s dS + \rho g \int_V (\nabla x \mathbf{\tilde{z}} \cdot \mathbf{v}) dV = 0, \tag{12}
\]

for all \(v \in \mathcal{X}\). The problem (12) can be associated to the following minimization problem:

\[
\mathcal{J}(\mathbf{u}) = \min \{ \mathcal{J}(v), \quad v \in \mathcal{X} \}, \tag{13}
\]

where

\[
\mathcal{J}(v) = \int_V B[D(v)]^{p-2}(D(v), D(v))_s dV + \int_{\Gamma_m} C|v|^{p-2}(|v|_s dS + \rho g \int_V (\nabla x \mathbf{\tilde{z}} \cdot \mathbf{v}) dV, \tag{14}
\]

which is strongly continuous, strictly convex and coercive on \(\mathcal{X}\) \cite{15,17}. Consequently, there exists a unique minimizer of \(\mathcal{J}\) in \(\mathcal{X}\), and equivalently, a unique solution to variational problem (12). In addition, setting \(v = \mathbf{u}\) in (12), using Korn and Poincaré’s inequalities, we obtain the stability inequality:

\[
\|\nabla \mathbf{u}\|_{L^p} \leq c \left( \frac{\rho g}{B} \|
abla x \mathbf{\tilde{z}}\|_{L^q} \right)^{\frac{1}{p-1}}, \tag{15}
\]

where \(q\) is defined by \(1/p + 1/q = 1\).

Finally, the SSA \cite{13,12} is obtained from the FOA by replacing the space of solutions \(\mathcal{X}\) by its subspace of vertically-constant functions:

\[
\mathcal{X}_S = \{ v \in \mathcal{X}, \quad \partial_z v = 0 \} \subset \mathcal{X}.
\]

As a consequence, the variational formulation for the SSA writes:

Find \(\mathbf{u}_S \in \mathcal{X}_S\) satisfying (12) for all \(v \in \mathcal{X}_S\). \tag{16}

Since the space \(\mathcal{X}_S\) is closed and a convex subset of \(\mathcal{X}\), \(\mathcal{J}\) admits a unique minimum \(\mathbf{u}_S\) in \(\mathcal{X}_S\), which is the unique solution of (16), see also \cite{16}. In addition, the stability inequality (15) remains valid with \(\mathbf{u}_S\) instead of \(\mathbf{u}\).
4 Estimator

Before looking for estimating the quantity $\|u - u_s\|$, we recall a few lemma that will be useful later.

4.1 Preliminary lemma

The first lemma consists of inequalities (see Lemma 2.1 in [1], slightly modified with the triangle inequality $|\xi| + |\eta| \leq 2(|\xi| + |\xi - \eta|) \leq 4(|\xi| + |\eta|)):

**Lemma 1** There exists $M_1, M_2 > 0$ such that:

$$|||\xi|||_{p-2} - |||\eta|||_{p-2} \leq M_1 (|\xi| + |\xi - \eta|),$$

$$|||\xi|||_{p-2} - |||\eta|||_{p-2} \leq M_2 \left(\frac{|\xi|^{p-2} - |\eta|^{p-2}}{2^p - 2}\right)(\xi - \eta),$$

for all $(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$.

The second lemma introduces the concept of solution-dependent quasi-norm (see in [1]):

**Lemma 2** Let $u$ be the solution of (12), then, the following mapping

$$v \mapsto |||v|||_u := \int_V B((D(u)) + |D(v)|_s)^{p-2}|D(v)|^2 dV + \int_{\Gamma} C ([|u| + |v|]^{p-2})dS,$$

defines a quasi-norm in $X$, i.e. it satisfies all properties of the norm except homogeneity. In addition, using Hölder's inequality, one can show there exist $L_1, L_2 > 0$ such that, for all $v \in X$, we have:

$$L_1 |||v|||_u^{2/p} \leq |v|_{W^{1,p}} \leq L_2 \left[|u|_{W^{1,p}} + |v|_{W^{1,p}}\right]^{2-p} |||v|||_u.$$  (20)

The last lemma states approximation properties of $X$ in $X$.

**Lemma 3** The following inequalities hold for all $v \in X$:

$$\left\| \frac{v - \nabla}{h} \right\|_{L^p(V)} \leq \|\partial_z v\|_{L^p(V)},$$

$$\left\| \frac{v - \nabla}{h^{1/q}} \right\|_{L^p(\Gamma)} \leq \|\partial_z v\|_{L^p(V)},$$

for $\Gamma = \Gamma_m, \Gamma_s$, where

$$\nabla(z) = \frac{1}{h(z)} \int_0^z v(\cdot, z)dz,$$

is a projector from $X$ to $\Gamma$. 
Proof: By integrating $v(x, z) - v(x, \xi) = \int_{\xi}^{z} \partial_z v(x, \zeta) d\zeta$ in $\xi$ from $z(x)$ to $\pi(x)$, we obtain

$$h(x) (v(x, z) - v(x)) = \int_{z(x)}^{\pi(x)} \int_{z(x)}^{\xi} \partial_z v(x, \zeta) d\zeta d\xi.$$ 

Then,

$$|v(x, z) - \nabla(x)| \leq \frac{1}{h(x)} \int_{z(x)}^{\pi(x)} \int_{z(x)}^{\xi} |\partial_z v(x, \zeta)| d\zeta d\xi \leq \int_{z(x)}^{\pi(x)} |\partial_z v(x, \zeta)| d\zeta.$$

By Hölder’s inequality, we have

$$|v(x, z) - \nabla(x)|^p \leq \left( \int_{z(x)}^{\pi(x)} |\partial_z v(x, \zeta)| d\zeta \right)^p = \left( \int_{z(x)}^{\pi(x)} |\partial_z v(x, \zeta)|^p d\zeta \right) \left( h(x) \right)^{\frac{q}{p}}.$$

Finally, we integrate in $z$:

$$\int_{z(x)}^{\pi(x)} |v(x, z) - \nabla(x)|^p dz \leq h(x) \left( h(x) \right)^{\frac{q}{p}} \int_{z(x)}^{\pi(x)} |\partial_z v(x, \zeta)|^p d\zeta = (h(x))^p \int_{z(x)}^{\pi(x)} |\partial_z v(x, \zeta)|^p d\zeta$$

and obtain (21) after dividing by $(h(x))^p$, bringing it in the left integral and integrating in $x$. On the other hand, we take (24) with $z = \pi(x)$, divide by $(h(x))^\frac{q}{p}$, and integrate over the whole horizontal domain, so that:

$$\int_{\Gamma} \left| \frac{v - \nabla}{h^{\frac{q}{p}}} \right|^p \leq \int_V |\partial_z v|^p,$$

which is exactly (22) for $\Gamma = \Gamma_s$. Using the same method with $z = \bar{z}(x)$ instead of $z = \pi(x)$, we can prove (22) for $\Gamma = \Gamma_m$ since $v(x, \cdot) = 0$ if $(x, \bar{z}(x)) \in \Gamma_0$. □

4.2 A priori estimator

Following [1,9,5] and using (20), one can show the Cea’s lemma [4] below.

Lemma 4 Let $u$ be the FOA solution of (12) and $u_S$ be the SSA solution of (16). Then, for all $v_S \in \mathcal{X}$ :

$$|||u - u_S|||_u \leq c|||u - v_S|||_u. \quad (26)$$

It now remains to transform the estimate (26) into an estimate with a standard norm. Using (20) and (26), we can show the following lemma.
Lemma 5 Let $u$ be the FOA solution of (12) and $u_S$ be the SSA solution of (16). Then, for all $v_S \in X$:

$$c\|u - u_S\|_{W^{1,p}}^2 \leq \left(\|u\|_{W^{1,p}} + \|u - v_S\|_{W^{1,p}}\right)^{2-p} \|u - v_S\|_{W^{1,p}}^p. \quad (27)$$

By choosing $v_S$ the vertical mean $u$, and using an approximation inequality for $X$ (Lemma 3), we could obtain an a priori error estimate, i.e. depending on the solution $u$. Instead, we prefer to derive an a posteriori estimator in the next section by a residual technique [6].

4.3 A posteriori estimator

We now state the main Theorem of the paper.

**Theorem 1** Let $u$ be the FOA solution of (12) and $u_S$ be the SSA solution of (16). Then, the following estimation holds:

$$\frac{|u - u_S|_{W^{1,p}}}{|u_S|_{W^{1,p}}} \leq c\left[\frac{2\rho g}{B}\|\nabla \tilde{x} \|_{L^q}\right]^{\frac{2-p}{2}} \left(\eta_1 + \eta_2 + \eta_3\right) \frac{|u|_{W^{1,p}}}{|u_S|_{W^{1,p}}}, \quad (28)$$

where

$$\eta_1 := \|h \times \left(-B \nabla \cdot \left(D(u_S)\right)^{p-2} D(u_S) + \text{tr}(D(u_S)I)\right) + \rho g \nabla \tilde{x}\|_{L^4(V)},$$

$$\eta_2 := \|h^{1/q} \times \left(B\|D(u_S)\|^{p-2} D(u_S) + \text{tr}(D(u_S)I)\right) \cdot n + C\|u_S\|^{q-2} \|u_S\|_{L^q(V)},$$

$$\eta_3 := \|h^{1/q} \times \left(B\|D(u_S)\|^{p-2} D(u_S) + \text{tr}(D(u_S)I)\right) \cdot n\|_{L^q(V)}.$$

**Proof:** By definition of the quasi-norm (19), and inequality (18), we have:

$$||| u - u_S |||_u \leq M_2 \int_V B \left(|D(u)|^{p-2} D(u) - |D(u_S)|^{p-2} D(u_S), D(u - u_S)\right)_s dV$$

$$+ M_2 \int_{\Gamma_m} C \left(|u|^{q-2} u - |u_S|^{q-2} u_S\right) \cdot (u - u_S) dS. \quad (29)$$

Since $u$ solves (12), then taking $v = u - u_S$, we deduce:

$$||| u - u_S |||_u \leq -M_2 \left(\int_V B|D(u_S)|^{p-2} (D(u_S), D(u - u_S))_s dV$$

$$- \int_{\Gamma_m} C(|u_S|^{q-2} u_S) \cdot (u - u_S) dS - \rho g \int_V \nabla \tilde{x} \cdot (u - u_S) dV\right).$$

Since $u_S$ solves (16), where $v_S \in X$ is arbitrary, we obtain:

$$||| u - u_S |||_u \leq -M_2 \left(\int_V B|D(u_S)|^{p-2} (D(u_S), D(u - u_S - v_S))_s dV$$

$$- \int_{\Gamma_m} C(|u_S|^{q-2} u_S) \cdot (u - u_S - v_S) dS - \rho g \int_V \nabla \tilde{x} \cdot (u - u_S - v_S) dV\right).$$
which becomes after using Gauß’s Theorem:

\[
\begin{align*}
||u - u_S||_u & \\
\leq -M_2 \left( \int_V \left[ -B \nabla \cdot \left( [D(u_S)]^{p-2}_* [D(u_S) + (D(u_S)) I] \right) + \rho g \nabla \kappa^3 \right] \cdot (u - u_S - v_S) \, dV \\
& \quad - \int_{\Gamma_m} \left( B \left[ [D(u_S)]^{p-2}_* [D(u_S) + \text{tr}(D(u_S)) I] \cdot n + C [u_S]^{p-2_*} (u_S) \right] \cdot (u - u_S - v_S) \, dS \\
& \quad - \int_{\Gamma_s} \left( B \left[ [D(u_S)]^{p-2}_* [D(u_S) + \text{tr}(D(u_S)) I] \cdot n \right] \cdot (u - u_S - v_S) \, dS \right) .
\end{align*}
\]

By Hölder’s inequality, we obtain

\[
\begin{align*}
||u - u_S||_u & \leq M_2 \left( \eta_1 \left\| \frac{u - u_S - v_S}{h} \right\|_{L^p(V)} + \eta_2 \left\| \frac{u - u_S - v_S}{h^{1/q}} \right\|_{L^p(\Gamma_m)} \\
& \quad + \eta_3 \left\| \frac{u - u_S - v_S}{h^{1/q}} \right\|_{L^p(\Gamma_s)} \right) .
\end{align*}
\]

Using the approximation inequalities of \( \mathcal{A} \) in \( \mathcal{X} \), namely (21) and (22) with \( v = u - u_S \) and taking \( v_S = \bar{v} \), we finally obtain

\[
\begin{align*}
||u - u_S||_u & \leq M_2 (\eta_1 + \eta_2 + \eta_3) ||\partial_z (u - u_S)||_{L^p(V)} \\
& \quad \leq M_2 (\eta_1 + \eta_2 + \eta_3) ||u - u_S||_{W^{1,p}(V)} .
\end{align*}
\]

In addition, (15) and (20) lead to:

\[
\begin{align*}
||u - u_S||_{W^{1,p}(V)} & \leq L_2 e^{2-p} \left[ \frac{2 \rho g \|\nabla \kappa^3\|_{L^p}}{B} \right]^{\frac{2-p}{p}} \left\| \frac{u - u_S}{h^{1/q}} \right\|_{L^p} .
\end{align*}
\]

The final result (28) follows from (30) and (31). \( \square \)

Let us note that the estimator (28) of Theorem 1 mostly depends on \( \eta_1 \), \( \eta_2 \) and \( \eta_3 \), which consist of residuals evaluated in \( u_S \) of the strong FOA equation (4), the free surface boundary condition (7) and the sliding boundary condition (9), respectively. As a matter of fact, those quantities would vanish if there were evaluated in the FOA solution \( u \) instead of the SSA solution \( u_S \). As a consequence, evaluating the residuals \( \eta_1 \), \( \eta_2 \) and \( \eta_3 \) in \( u_S \) provides a measure of the misfit between FOA and SSA.

5 Numerical results

ISMIP-HOM [14] experiments consist of modelling exercises based on various ice geometries and boundary conditions in order to generate different types of ice flow, which can be met in real glacier and ice sheet modelling. Here, we utilize the solutions of experiment D [14] computed with the FOA and the SSA in order to evaluate the capabilities of the estimator proved in Theorem 1 to assess the mismatch between the two solutions. For this purpose, one runs ISMIP-HOM experiment D since it represents a wide range of sliding-dominant flows when varying the aspect ratio of the ice domain.

The goal of ISMIP-HOM experiment D is to compute the velocity fields for a given simple domain of ice, boundary conditions, and parameters. More precisely,
we use the following parameters: \( p = \frac{4}{3} \) and \( B = 8.5768 \times 10^7 \) Pa s^{1/3}, while the ice geometry is defined by a parallel slab:

\[
\pi(x) = -x \tan(0.1^\circ), \quad \underline{a}(x) = \pi(x) - 1000, \quad h(x) = 1000, \quad x \in [0, X],
\]

and a slip condition is prescribed everywhere on the lower surface defined by \( s = 2 \) (note that choosing \( s = 2 \) does not fulfil the theoretical requirement \( s \in (1, p] \) since \( p = 4/3 \)) and

\[
C(x) = 1000 \times (1 + \sin(2\pi x/X)),
\]

where \( X > 0 \) is the length of the horizontal domain. Periodic boundary conditions connect the left- and right-hand sides of \( \Omega \), and a stress-free condition is prescribed on the top surface, see [14] for further details. Since there is no lateral (in \( y \)) variation, two-dimensional models are used to compute the solutions.

In order to assess the efficiency of estimator (28) in Theorem 1, we compute the FOA and SSA model solutions. For that, we implement the Newton multigrid solver described in [10]. In particular, we use a 2D mesh extruded from a subdivision of the horizontal segment \( \Omega \) into 64 equal sized segments with 16 uniformly distributed layers for the FOA and a single layer for the SSA. In order to simulate a wide range of aspect ratios \( \epsilon := h/X \), we run the ISMIP D experiment for several lengths of domain \( X \): \( X = 5, 10, 20, 40, 80 \) and 160 km. Fig. 2 (a,b,c) shows the FOA and SSA surface horizontal velocities for \( X = 10, 40, \) and 160 km. In addition, Fig. 2 (d) displays the normalized error (left-hand side of (28)) and its estimator (right-hand side of (28)) of Theorem 1 with respect to the aspect ratio \( \epsilon \).

Fig. 2 a,b,c: FOA and SSA surface horizontal velocities for ISMIP-HOM experiment D with \( X = 10 \) (a), 40 (b) and 160 km (c). d: Normalized error (left-hand side of (28)) versus its estimator (right-hand side of (28)) for several aspect ratios.
As expected, Fig. 2 (a,b,c) shows that the SSA solution converges to the FOA one when the aspect ratio $\epsilon$ shrinks to zero. Indeed, the approximation by vertically-constant functions gets globally better on shallow domains. As a matter of fact, Figure 2 d shows that both the error and its estimator decrease non-linearly, but similarly to zero. This validates the estimator (28) proved in Theorem 1.

6 Conclusions

In the literature, the Shallow Shelf Approximation (SSA) model is often introduced as a mechanical approximation of the First-Order Approximation (FOA). However, the SSA can also be seen as a numerical approximation by vertically-constant functions. By adopting this point of view, we derived an estimator, which can predict the accuracy of the SSA against the FOA, by following residual a posteriori error estimator techniques. The reliability of the estimator was validated for a relevant case of the SSA, which involves sliding-dominant flows as prevailing over ice shelves. In particular, we proved that our estimate can predict the non trivial decay of the discrepancy between the FOA and the SSA when decreasing the aspect ratio. It is hoped that this computable estimator will help modellers to assign suitable ice flow models (SSA, FOA, or a spatial combination of both) in an optimal way.

References