Multilayer shallow shelf approximation: minimisation formulation, finite element solvers and applications

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Abstract

In this paper, a multilayer generalisation of the Shallow Shelf Approximation (SSA) is considered. In this recent hybrid ice flow model, the ice thickness is divided into thin layers, which can spread out, contract and slide over each other in such a way that the velocity profile is layer-wise constant. Like the SSA (1-layer model), the multilayer model can be reformulated as a minimisation problem. However, unlike the SSA, the functional to be minimised involves a new penalisation term for the interlayer jumps of the velocity, which represents the vertical shear stresses induced by interlayer sliding. Taking advantage of this reformulation, numerical solvers developed for the SSA can be naturally extended layer-wise or column-wise. Numerical results show that the column-wise extension of a Newton multigrid solver proves to be robust in the sense that its convergence is barely influenced by the number of layers and the type of ice flow. In addition, the multilayer formulation appears to be naturally better conditioned than the one of the first-order approximation to face the anisotropic conditions of the sliding-dominant ice flow of ISMIP-HOM experiments.

1 Introduction

Glaciologists need ice flow models which can be run at very large scales (in space and time) and treat the mechanics adequately while being computationally tractable. Examples of applications are in marine ice sheet modelling in order to better evaluate sea level rise in a climate change regime [34], or in paleoglaciology in order to reconstruct the extents of glaciers and ice sheets over glacial cycles [25]. Despite some recent progress achieved in parallelising solvers [2] or domain decomposition [31], 3D models remain too computationally demanding to be used for that purpose. In addition, remeshing procedures in 3D are complex to apply, since meshes must conserve a certain quality in order to preserve the performances of the solver [16]. For this reason, simplified zero-order models of reduced complexity like the Shallow Ice Approximation (SIA) [12] or the Shallow Shelf Approximation (SSA) [18, 17, 35] are still popular in the community of glacier and ice sheet modellers for running large-scale simulations. Based on the assumption of small aspect ratios of the ice geometries, the SIA, which is a mathematical 1D (vertical) model, accounts only for vertical shear stresses, while the SSA, which is mathematically 2D (horizontal), accounts only for longitudinal stresses. If it is justified to use either the SSA or the SIA in some localised parts of the ice domain, it is often necessary to combine the two when modelling the ice flows of an entire glacier or ice sheet. For example, the vertical shear components

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of the stress tensor are significant where ice is frozen to the ground, while the longitudinal components are dominant in the fast-sliding parts, like the floating areas, such that using the SIA on the grounded part and the SSA on the floating area might look acceptable. Unfortunately, such an approach is not suitable in the vicinity of the Grounding Line (GL) which delimits the grounded and the floating areas, as all components must be accounted for [29]. This has driven the construction of “hybrid” models, which account for both kind of stresses, while being mathematically 2D. The simplest hybrid model consists of the linear combination SIA+SSA, which is arrived at by adding together the velocities of each model [3]. Unfortunately, this model does not include the simultaneous coupling between the vertical shear and the longitudinal stresses. As a result, the model cannot capture the 3D ice flows that occur in deep and narrow valleys or in the vicinity of GLs [23, 14].

In contrast, the L1L2 [11] or some variants like the ones proposed in [24], [30] or in [8] include the vertical shear stress in the computation of the effective viscosity of the SSA. All these hybrid models have in common that they solve a single non-linear elliptic 2D problem, and that the velocity profile is reconstructed \textit{a posteriori} via an implicit relation [30, 36, 4].

In this paper, a recently introduced hybrid multilayer model generalising the SSA is considered [13]. This approach consists of seeing the ice thickness as a pile of thin layers which can spread out, contract and slide over each other. Assuming a vertically piecewise-constant velocity profile in each layer, the model derives from local depth-integrations of the hydrostatic approximation [1, 20]. The crucial step when deriving the model consists of redefining the interlayer tractions by keeping only the shear components. The final multilayer model consists of a tridiagonal system of 2D non-linear elliptic equations, whose number corresponds to the number of layers. By construction, this multilayer model naturally generalises the SSA, which corresponds to the 1-layer case of the model. As a consequence, it is called “Multilayer Shallow Shelf Approximation” in this paper and is abbreviated as MSSA. Unlike the SSA, the MSSA is hybrid since it combines the longitudinal and the vertical shear stresses. Like the SSA, the MSSA model can be reformulated as a minimisation problem for a certain functional. Interestingly, with such a reformulation, the new term corresponding to vertical shear stresses can been interpreted as a penalisation term for the interlayer jumps of the velocity components. Finally, in contrast to the First-Order Approximation (FOA) [1, 20] which consists of a 3D elliptic problem or the Stokes model [6, 15] which consists of a 3D saddle-point problem, the MSSA consists of a system of 2D elliptic equations, and thus is much easier to solve. Moreover, any solver that has been developed for the SSA can be extended to solve the MSSA layer-wise or column-wise. The performance of the resulting numerical solvers is tested for the prognostic benchmark flow-line experiments B and D of the ISMIP-HOM project [22]. In addition, one applies the MSSA model to the first test problem proposed by the Marine Ice Sheet Model Inter-comparison Project (MISMIP) [21].

This paper is organised as follows: in Section 2, the SSA model and its multilayer extension are first recalled. Then the MSSA model is reformulated as a minimisation problem. Afterwards, two numerical methods based on a SSA solver are described for solving the MSSA system in Section 3. Lastly, numerical results are reported in Section 4.

2 Model

In this section, a generic two- and three-dimensional system of ice sheet and ice shelf is considered. For the three-dimensional model \((d = 3)\), the ice sheet extends over a two-
dimensional horizontal domain contained in $\Omega \subset \mathbb{R}^2$. Its height and all other quantities will be described as functions over $\Omega$. If we assume that no physical variable varies in the horizontal direction $y$, then a three-dimensional ice sheet can be described by a single vertical section at $y = 0$, leading to a two-dimensional ice sheet model ($d = 2$), see Fig. 1. In this model the ice sheet extends over a one-dimensional horizontal domain contained in $\Omega \subset \mathbb{R}$ that is orthogonal to the direction $y$ of constant shape. Although such ice sheet is not physical, it is useful for the sake of understanding.

Following the notations introduced in [13], the domain of ice is defined by

$$\{ (x, z), \ x \in \Omega, \ s(x) \leq z \leq \bar{s}(x) \}, \quad (1)$$

where $\Omega \subset \mathbb{R}^{d-1}$ represents its horizontal projection, $x = x$ or $x = (x, y)$ denote the horizontal coordinates for $d = 2$ or 3, $z$ denotes the vertical coordinate, $b(x)$, $s(x)$ and $\bar{s}(x)$ are the elevations of the bedrock, the lower and upper ice surfaces; see Figure 1. Note that $s = b$ holds where ice is grounded and $s > b$ where ice is floating. The flotation of ice is driven by the Archimedes principle,

$$\bar{s} = \max \left\{ b, -\frac{\rho}{\rho_w} h \right\}, \quad (2)$$

where $h := \bar{s} - s$ is the ice thickness and $\rho$ and $\rho_w$ are the constant densities of ice and water, respectively (see Figure 1). Relation (2) says that if the buoyancy $-\rho_w gb$ is less than the ice overburden $\rho gh$, then ice is grounded, otherwise ice is floating and $\rho/\rho_w$ of the ice thickness is below sea level.

Figure 1: Cross-section of an ice sheet and an ice shelf, with notations.

At the lower interface, ice might be frozen to the ground, sliding on the ground or floating on water. In what follows, $\Omega_0$, $\Omega_m$, $\Omega_f$ and $\Omega_l$ denote the projection into the horizontal plane of the non-sliding part, the sliding grounded part, the floating part and the calving front, respectively. As a matter of fact, $\Omega = \Omega_0 \cup \Omega_m \cup \Omega_f$ and $\Omega_l \subset \partial \Omega$. In this paper $\nabla = \nabla_x$, $\nabla \cdot = \nabla \cdot_x$ denote gradient and divergence operators, respectively, with respect to the horizontal variables $x$.

Section 2 is organised as follows. First, the commonly used SSA model [18, 17, 35, 14, 26] is recalled in Section 2.1. Second, the MSSA system derived in [13] is recalled in a vectorial form and reformulated as variational and minimisation problems in Section 2.2.
2.1 Shallow Shelf Approximation (SSA)

Only in Section 2.1, sliding is assumed to occur everywhere on the grounded part of the ice domain such that the no-sliding area is empty: $\Omega_0 = \emptyset$. The derivation of the SSA relies on the hydrostatic approximation and the assumption of a vertically constant velocity profile [9]. As a result, the horizontal velocity field $u(x) \in \mathbb{R}^{d-1}$ is determined by [14, 26]:

$$-A^{-\frac{1}{n}} \nabla \cdot \left( h \left| D(u) \right|^{\frac{1}{n}-1} [D(u) + \text{tr}(D(u))I] \right) + \tau_c = -\rho gh \nabla \sigma,$$

(3)

where $A > 0$ and $n \geq 1$ are two coefficients called the rate factor and Glen’s exponent, $D(u) := \frac{1}{2} (\nabla u + \nabla u^T)$ denotes the strain rate of $u$, $\text{tr}$ is the trace operator, $I$ the identity second-order tensor and $|Y|_\ast := \sqrt{\langle Y, Y \rangle_\ast}$ associated with the scalar product defined by

$$(Y, Z)_\ast := \frac{1}{2} (\text{tr}(YZ) + \text{tr}(Y)\text{tr}(Z)),$$

for all pairs of $2 \times 2$ matrices $(Y, Z)$, and $\tau_c$ is the basal traction. In reality, $A$ is not constant since it depends on ice temperature [7, 19]. However, for the sake of simplicity, it is assumed in this paper that the ice is isothermal. Heuristically, the first term in (3) represents the longitudinal stresses, the second term represents the friction on the bedrock and the right-hand-side represents the driving stress. In many applications of the SSA, the friction is governed by Weertman’s law [12], such that the term $\tau_c$ in (3) is replaced by

$$\tau_c = C |u|^{\frac{1}{m}-1} u \times 1_{\Omega_m},$$

(4)

where $C = C(x) > 0$ is prescribed, $m > 0$ is a given parameter, and $1_R(x)$ equals 1 if $x \in R$ and 0 otherwise. Conditions on the boundary of the domain of ice $\partial \Omega$ can be written [14]

$$A^{-\frac{1}{n}} h \left| D(u) \right|^{\frac{1}{n}-1} [D(u) + \text{tr}(D(u))I] \cdot n = F_n,$$

(5)

where

$$F = \frac{1}{2} \rho gh^2 - \frac{1}{2} \rho \omega g (\min\{s, 0\})^2,$$

(6)

and $n$ denotes a horizontal outward normal vector to $\Omega$. At the boundary of the ice domain $\partial \Omega$, the ice thickness $h$ is zero except at the calving front $\Omega_l$ where ice cliffs might occur. At the calving front $\Omega_l$, condition (5) says that the outward pressure of ice is partially balanced by the hydrostatic sea water pressure [26], while elsewhere, $h = 0$ simply implies $F = 0$ in (5).

Multiplying (3) by a test function $v$, integrating over $\Omega$, using the divergence theorem, the boundary condition (5) and the equality [26]

$$[D(u) + \text{tr}(D(u))I] \cdot \nabla v = (D(u), D(v))_\ast,$$

(7)

lead to the variational equality

$$A^{-\frac{1}{n}} \int_\Omega h \left| D(u) \right|^{\frac{1}{n}-1} (D(u), D(v))_\ast d\Omega + \int_{\Omega_m} C |u|^{\frac{1}{m}-1} u \cdot v d\Omega + \rho g \int_\Omega h \nabla \sigma \cdot v d\Omega - \int_{\partial \Omega} F_n \cdot v dS = 0.$$  

(8)
One can verify that (8) is the Euler-Lagrange equation, \( \langle D \mathcal{J}(u), v \rangle = 0 \), for the functional

\[
\mathcal{J}(u) := \frac{A}{1} + \frac{1}{\alpha} \int_{\Omega} h |D(u)|_{\frac{1}{\alpha} + 1} d\Omega + \frac{1}{\beta} + \int_{\Omega_m} C |u|_{\frac{1}{\beta} + 1} d\Omega \\
+ \rho g \int_{\Omega} h \nabla \bar{z} \cdot u d\Omega - \int_{\partial\Omega} F n \cdot u dS.
\] (9)

More precisely, one can show that solving (8) is equivalent to solving the minimisation problem:

Find \( u \) s.t. \( \mathcal{J}(u) \leq \mathcal{J}(v) \), \( \forall v \). (10)

Since \( \mathcal{J} \) is convex, strongly continuous in \([W^{1,1+(\frac{1}{\alpha}, (\Omega)}]^{d-1}\) and coercive if \( h \) is uniformly lower-bounded by a positive constant [26, 28], there exists a minimiser of \( \mathcal{J} \), provided this last assumption is satisfied. In addition, \( \mathcal{J} \) is strictly convex and the minimiser is unique if \( \Omega_m \) has a positive measure.

### 2.2 Multilayer Shallow Shelf Approximation (MSSA)

Following [13], the domain of ice is now divided in the vertical direction into \( L \) layers of thickness \( h_1, \ldots, h_L \) such that

\[
\sum_{l=1,\ldots,L} h_l = h, \tag{11}
\]

see Figure 2. Call \( s^l = \bar{z} + \sum_{j=1}^{l} h^j \) the elevation of the upper surface of layer \( l \) for \( l = 1, \ldots, L \).

![Figure 2: Multilayer splitting of the ice thickness.](image)

The derivation of the SSA model (Section 2.1) is based on the assumption of a constant velocity profile [18, 17, 35]. In contrast, the horizontal velocity is assumed here to be vertically piecewise-constant [13], equal to \( u^l \in \mathbb{R}^{d-1} \) on layer \( l \):

\[
u(x, z) = \sum_{l=1,\ldots,L} u^l(x) 1_{(s^{l-1}, s^l)}(z), \tag{12}
\]
where $1_I(z)$ equals 1 if $z \in I$ and 0 otherwise, see Figure 2. For the sake of convenience, the derivation of the MSSA is briefly recalled, however, all the details can be found in [13]. Integrating vertically the hydrostatic approximation of the Stokes equations over layer $l$ yields to:

$$-A^{-\frac{1}{n}} \nabla \cdot \left( h^l \left| D\left(u^l\right) \right|^{\frac{1}{n}-1} \left[ D\left(u^l\right) + \text{tr}(D(u^l))I \right] \right) - S^l + S^{l-1} = -\rho g h^l \nabla \pi, \quad (13)$$

where $S^l$ and $-S^{l-1}$ are the interlayer tractions, at the upper and the lower interfaces, respectively. Because of the discontinuity of the velocity field across the layer interfaces (Equation (12)), $S^l$ and $-S^{l-1}$ are undefined. Two types of redefinition of the interlayer tractions were proposed in [13]. First, they can be redefined at zeroth-order in the interlayer surface slope by keeping only the vertical shear stresses components. Second, if the layers are chosen such that there are aligned with the streamlines, then, second-order accurate interlayer tractions can be advantageously used instead of zeroth-order ones. Of course, in practice, the streamlines are never known a priori, however, an empirical estimate can be used instead, leading to an improved model, see [13]. In this paper, only the most simple zeroth-order interlayer tractions are considered for simplicity, and because using the second-order ones instead of the zeroth-order ones does not affect the numerical performances of Section 4.1 neither improve substantially the mechanical performances of the model in the case of the marine ice sheet considered in Section 4.2. For convenience, the variational and minimisation forms of the multilayer model including the second-order accurate interlayer tractions are written in A.

As a matter of fact, the approximation of the stress tensor (14) is similar to one of the Shallow Ice Approximation [12, 9]. In the same way, the conditions on the bedrock (no-slip on $\Omega_0$, sliding with Weertman’s law [12] on $\Omega_m$ and floating on $\Omega_f$) can be summarised [13] by

$$S^0 = A^{-\frac{1}{n}} \left| \frac{u^1}{h^1} \right|^{\frac{1}{n}-1} \left( \frac{u^1}{h^1} \right) \times 1_{\Omega_0} + C \left| u^1 \right|^{\frac{1}{n}-1} \left( u^1 \right) \times 1_{\Omega_m}, \quad (15)$$

while the free boundary condition on the upper surface rewrites $S^L = 0$. It follows that the MSSA solution $(u^1, ..., u^L)$ solves the following $d - 1 \times d - 1$-block tridiagonal system of two-dimensional non-linear elliptic equations [13]:

$$-A^{-\frac{1}{n}} \nabla \cdot \left( h^L \left| D\left(u^L\right) \right|^{\frac{1}{n}-1} \left[ D\left(u^L\right) + \text{tr}(D(u^L))I \right] \right) + A^{-\frac{1}{n}} \left| \frac{u^L - u^{L-1}}{h^L + h^{L-1}} \right|^{\frac{1}{n}-1} \left( \frac{u^L - u^{L-1}}{h^L + h^{L-1}} \right) = -\rho g h^L \nabla \pi, \quad (16)$$
for all \( l \in \{2, \ldots, L-1\} \):

\[
-A^{-\frac{1}{2}} \nabla \cdot \left( h^l \left| D(u^l) \right|^{\frac{1}{2} - 1} \left[ D(u^l) + \text{tr}(D(u^l))I \right] \right) + A^{-\frac{1}{2}} \frac{u^l - u^{l-1}}{h^l + h^{l-1}} \left| \frac{u^l - u^{l-1}}{h^l + h^{l-1}} \right|^{\frac{1}{2} - 1} \left( \frac{u^l - u^{l-1}}{h^l + h^{l-1}} \right) = -\rho g h^l \nabla s,
\]

and

\[
-A^{-\frac{1}{2}} \nabla \cdot \left( h^1 \left| D(u^1) \right|^{\frac{1}{2} - 1} \left[ D(u^1) + \text{tr}(D(u^1))I \right] \right) + A^{-\frac{1}{2}} \frac{u^1 - u^2}{h^1 + h^2} \left| \frac{u^1 - u^2}{h^1 + h^2} \right|^{\frac{1}{2} - 1} \left( \frac{u^1 - u^2}{h^1 + h^2} \right) = -\rho g h^1 \nabla s.
\]

At the boundary of the ice domain \( \partial \Omega \), the MSSA formulation of (5) is

\[
A^{-\frac{1}{2}} h^l |D(u^l)|^{\frac{1}{2} - 1}[D(u^l) + \text{tr}(D(u^l))I] \cdot n := F^l n,
\]

where

\[
F^l = \frac{1}{2} \rho g \left[ (\bar{s} - s^l)^2 - (\bar{s} - s^{l+1})^2 \right] + \frac{1}{2} \rho g \left[ (\min(s^{l+1}, 0))^2 - (\min(s^l, 0))^2 \right],
\]

and \( n \) denotes a horizontal outward normal vector to \( \Omega_i \).

It is interesting to notice the similarity of the system (16) (17) (18) with the vectorial equation (3) of the SSA, which corresponds to the 1-layer model (i.e., when \( L = 1 \)). Indeed, the SSA model consists of a single elliptic non-linear equation while the MSSA consists of a system of elliptic non-linear equations. Unlike the SSA, the system (16) (17) (18) has additional terms, which couple the layers, and which represent the vertical shear stresses. In contrast with other hybrid models like the L1L2 [11] or the variants proposed in [24] or [30], the terms for the longitudinal and the vertical shear stresses are not embedded, but additionally decoupled in (16) (17) (18).

To analyse the MSSA system (16) (17) (18) and to implement a finite element method, it must be rewritten as a variational problem. Multiplying each equation of the system (16) (17) (18) by a test function \( v^l \), adding up and integrating over \( \Omega \) lead to

\[
-A^{-\frac{1}{2}} \sum_{l=1, \ldots, L} \int_{\Omega} \nabla \cdot \left( h^l \left| D(u^l) \right|^{\frac{1}{2} - 1} \left[ D(u^l) + \text{tr}(D(u^l))I \right] \right) \cdot v^l d\Omega + A^{-\frac{1}{2}} \int_{\Omega_0} \frac{u^1 - u^2}{h^1 + h^2} \left| \frac{u^1 - u^2}{h^1 + h^2} \right|^{\frac{1}{2} - 1} \left( \frac{u^1 - u^2}{h^1 + h^2} \right) \cdot v^1 d\Omega
\]

\[
+ A^{-\frac{1}{2}} \sum_{l=2, \ldots, L} \int_{\Omega} \frac{u^l - u^{l-1}}{h^l + h^{l-1}} \left| \frac{u^l - u^{l-1}}{h^l + h^{l-1}} \right|^{\frac{1}{2} - 1} \left( \frac{u^l - u^{l-1}}{h^l + h^{l-1}} \right) \cdot (v^l - v^{l-1}) d\Omega = -\rho g \int_{\Omega} \sum_{l=1, \ldots, L} h^l \nabla s \cdot v^l d\Omega.
\]
Using the divergence theorem, the boundary condition (19) and Equality (7) lead to the variational equality

\[
A^{-\frac{1}{n}} \sum_{l=1}^{L} \int_{\Omega} h^l |D(u^l)|^{\frac{1}{n} - 1} (D(u^l), D(v^l))_s d\Omega
\]

(22)

\[
+ A^{-\frac{1}{n}} \int_{\Omega_0} \left( \frac{1}{h^1} \right)^{\frac{1}{n}} |u^1|^{\frac{1}{n} + 1} d\Omega + \frac{1}{m + 1} \int_{\Omega_m} C |u^1|^{\frac{1}{n} + 1} d\Omega
\]

\[
+ A^{-\frac{1}{n}} \sum_{l=2}^{L} \int_{\Omega} \left( \frac{1}{h^l + h^{l-1}} \right)^{\frac{1}{n}} |u^l - u^{l-1}|^{\frac{1}{n} + 1} d\Omega
\]

\[
= -\rho g \int_{\Omega} \sum_{l=1}^{L} h^l \nabla \pi \cdot v^l d\Omega + \int_{\Omega} \sum_{l=1}^{L} F^l n \cdot v^l dS.
\]

Like in Section 2.1, one can verify that (22) is the Euler-Lagrange equation, \( \langle D\mathcal{J}(u), v \rangle = 0 \), where \( u = (u^1, ..., u^L) \) and \( v = (v^1, ..., v^L) \), for the functional

\[
\mathcal{J}(u) = A^{-\frac{1}{n}} \sum_{l=1}^{L} \int_{\Omega} h^l |D(u^l)|^{\frac{1}{n} + 1} d\Omega
\]

(23)

\[
+ A^{-\frac{1}{n}} \int_{\Omega_0} \left( \frac{1}{h^1} \right)^{\frac{1}{n}} |u^1|^{\frac{1}{n} + 1} d\Omega + \frac{1}{m + 1} \int_{\Omega_m} C |u^1|^{\frac{1}{n} + 1} d\Omega
\]

(24)

\[
+ A^{-\frac{1}{n}} \sum_{l=2}^{L} \int_{\Omega} \left( \frac{1}{h^l + h^{l-1}} \right)^{\frac{1}{n}} |u^l - u^{l-1}|^{\frac{1}{n} + 1} d\Omega
\]

(25)

\[
+ \rho g \int_{\Omega} \sum_{l=1}^{L} h^l \nabla \pi \cdot u^l d\Omega - \int_{\Omega} \sum_{l=1}^{L} F^l n \cdot u^l dS,
\]

(26)

and solving (22) is equivalent to solving the minimisation problem:

\[
\text{Find } u \text{ s.t. } \mathcal{J}(u) \leq \mathcal{J}(v), \quad \forall v.
\]

(27)

Like the SSA, minimisation problem (27) consists of a vectorial p-Laplace problem with \( p = 1 + \frac{1}{n} < 2 \). However, unlike the SSA, this vectorial p-Laplace problem sums with the interlayer terms (25), which couple the layers. While the p-Laplace term (23) corresponds to the longitudinal stresses, the term (25) corresponds to the vertical shear stress components. This last term can be seen as a penalisation of jumps for the piecewise-constant velocity profile, as in formulations of the Discontinuous Galerkin method [10]. Interestingly, this penalisation involves a special norm, which relies on Glen’s exponent \( n \) and the thickness of layers. In addition, both terms in (24) apply only on the lowest layer and correspond to the no-slip and the sliding conditions, respectively. Finally, the first term in (26) represents the driving stress while the second corresponds to the balance between the ice pressure and the hydrostatic sea water pressure at the calving front. For convenience, the MSSA system similar to (21), but based on second-order reconstructed interlayer tractions [13] instead of zeroth-order ones, is written in A together with its associated variational formulation and functional to minimize \( \mathcal{J} \).

As in Section 2.1 for the SSA, one can show that the functional \( \mathcal{J} \) is convex, strongly continuous in \( [W^{1,1+\frac{1}{n}}(\Omega)]^{(d-1)L} \) and therefore weakly lower semi-continuous. Additionally, \( \mathcal{J} \) is coercive if \( h^l \) is uniformly lower-bounded by a positive constant [26, 28]. This implies the existence of a minimiser of \( \mathcal{J} \), provided this last assumption is satisfied. In addition, \( \mathcal{J} \) is strictly convex and the minimiser is unique if \( \Omega_m \cup \Omega_0 \) has a positive measure.
It is interesting to notice that when the thickness of one layer gets small ($h^l \to 0$), the weakening of coerciveness of the $p$-Laplace term (23) is compensated by the strengthening of the penalisation terms (25), which is weighted by $(1/h^l)^{1/n}$. As a consequence, a very small (but non-zero) $h^l$ does not cause the problem (27) to degenerate as long as the entire thickness $h$ remains uniformly lower-bounded by a positive constant.

2.3 Mass conservation

The mass conservation principle in a vertical column of ice can be written as [9]:

$$\frac{\partial h}{\partial t} + \nabla \cdot \left( \int_{\mathbb{R}} \mathbf{u} \, dz \right) = a,$$

where $a$ is the yearly averaged positive or negative external ice mass balance due to melting and solid precipitation. Since $\mathbf{u}$ is layer-wise constant, the flux in (28) can be easily computed via

$$\int_{\mathbb{R}} \mathbf{u} \, dz = \sum_{l=1}^{\mathcal{L}} h^l \mathbf{u}^l.$$

3 Numerical methods to solve the MSSA equations

In this section, two strategies (later referred to as “layer-wise” and “column-wise”) are developed to extend any finite element solver for the traditional SSA to the MSSA system (16) (17) (18) the boundary condition (19). Like for the SSA [14], the variational problem (22), or equivalently the minimisation problem (27), are appropriate to implement the finite element method.

Consider a regular mesh of $\Omega \subset \mathbb{R}^{d-1}$ that is made of segments when $d = 2$ and of triangles when $d = 3$, and the finite element space, which is spanned by the continuous, linear functions on each element. Since this finite element space can be identified with $\mathbb{R}^{I \times d-1}$, where $I$ is the number of nodes of the mesh, the Ritz-Galerkin approximation [5] of the minimisation problem (27) reads:

Find $U \in \mathbb{R}^{\mathcal{L} \times I \times d-1}$ s.t. $\mathcal{J}_h(U) \leq \mathcal{J}_h(V)$, $\forall V \in \mathbb{R}^{\mathcal{L} \times I \times d-1}$,

where the vector $U$ contains the nodal values of the approximation of $\mathbf{u}$ and $\mathcal{J}_h : \mathbb{R}^{\mathcal{L} \times I \times d-1} \to \mathbb{R} \cup \{+\infty\}$ is strictly convex, coercive and lower semi-continuous. Here $\mathcal{J}_h$ is either $\mathcal{J}_h = \mathcal{J}$ defined by (23)-(26) or an approximation of $\mathcal{J}$ by numerical quadratures.

Finite element solvers for the SSA (i.e. (30) with $L = 1$) have been used and described in the literature from the simple non-linear Gauß–Seidel method [33], to the Newton Multigrid method presented in [14]. Later, one uses the acronym ‘GS’ and ‘NM’ to denote the Gauß–Seidel and Newton Multigrid solvers aforementioned. While the GS method consists of minimising successively $\mathcal{J}_h$ in each coordinate directions $(i,k) \in I \times d-1$, the NM one combines one GS step and a Newton-type acceleration with a linear geometric multigrid method for solving the correction step. Both methods have in common that they apply sequentially a scalar minimisation procedure for given indices $(i,k) \in I \times d-1$. In this paper, such solver is assumed to be available, and only its extension to the MSSA system (16) (17) (18) is described.

There are two natural ways to loop over the set of indices $(l,i,k) \in L \times I \times d-1$: layer-wise, i.e., looping first over the layer indices $l$ and then over the horizontal node indices $i$ or the other alternative, i.e column-wise. Opting for one strategy or the other leads to two different methods, which are described in turn in the next two sections.
3.1 Layer-wise extension method

In the layer-wise extension method, the approximation sequence

\[ U_\nu = (U_1^\nu, ..., U_L^\nu), \]

(where \( U_l^\nu \in \mathbb{R}^{I \times d-1} \)), which is initialised by \( U_0 \), is defined recursively by taking the solutions of the successive minimisation problems:

\[ J (U_{\nu+1}^1, U^2_\nu, ..., U_L^\nu) \leq J (V, U^2_\nu, ..., U_L^\nu), \quad \forall V, \]

\[ J (... , U_{\nu+1}^{l-1}, U_{\nu+1}^{l}, U_{\nu+1}^{l+1}, ...) \leq J (... , U_{\nu+1}^{l-1}, V, U_{\nu+1}^{l+1}, ...), \quad \forall V, \]

\[ J (U_{\nu+1}^1, U^2_{\nu+1}, ..., U_L^\nu) \leq J (U_{\nu+1}^1, U^2_{\nu+1}, ..., V), \quad \forall V, \]

where \( V \) is taken in \( \mathbb{R}^{I \times d-1} \). Thus, this first method consists of solving the MSSA system (16), (17) or (18) layer by layer, from the lowest one \( l = 1 \) to the highest one \( l = L \), using the old solution \( U_\nu \) in the interface term (25), and to iterate. Finally any SSA-type solver (like GS or NM) can be used to solve each individual minimisation problem. However, the convergence of this method severely deteriorates when the number of layers grows, as indicated in the numerical results of Section 4.1. Computing the MSSA solution for the “infinite parallel-sided slab” [13] even shows (not displayed) that the number of iterations needed to obtain a given accuracy increases exponentially with the number of layers. This can be justified as follows: further efforts are needed to transfer a certain amount of information between the lowest and the highest layers if there are many layers. This phenomenon is even more pronounced if the vertical coupling is strong, as confirmed in Section 4.1. As a consequence, this method is not efficient for treating general ice flows, e.g., those in which the shear stresses significantly enhances the vertical coupling. Except for sliding-dominant shallow ice flows, it is recommended to use the second method, which is described in the next section.

3.2 Column-wise extension method

In contrast to the first one, the second method initially loops over the node indices \( i \) before looping over layers \( l \). For instance, if one opts for GS as SSA solver, the sequence \( U_\nu = (U_1^\nu, ..., U_I^\nu) \), (where \( U_i^\nu \in \mathbb{R}^{L \times d-1} \)), which is initialised by \( U_0 \), is defined recursively by taking the solutions of the successive minimisation problems

\[ J (U_{\nu+1}^1, U^2_{\nu+1}, ..., U_I^\nu) \leq J (V, U^2_{\nu+1}, ..., U_I^\nu), \quad \forall V, \]

\[ J (... , U_{\nu+1}^{i-1}, U_{\nu+1}^{i}, U_{\nu+1}^{i+1}, ...) \leq J (... , U_{\nu+1}^{i-1}, V, U_{\nu+1}^{i+1}, ...), \quad \forall V, \]

\[ J (U_{\nu+1}^1, U^2_{\nu+1}, ..., U_I^\nu) \leq J (U_{\nu+1}^1, U^2_{\nu+1}, ..., V), \quad \forall V, \]

where \( V \) is taken in \( \mathbb{R}^{L \times d-1} \). As said above, the GS method written above can be better replaced by the NM method described in [14]. In any case, a method still needs to be defined to solve each column-wise minimisation problem (of size \( \mathbb{R}^{L \times d-1} \)). For that, a non-linear block Gauß-Seidel method, which minimises successively from the lowest to the highest layer, is used. As expected, this column-wise extension method converges in practice much faster compared to the layer-wise extension method, in particular, when solving strongly vertically coupled ice flows, see Section 4.1.
3.3 Nested initialisation strategy

In practice, it might take a lot of iterations to reach the final solution when initializing $U_0$ by zero, see Section 4.1. To achieve a better initialisation, one can apply a nested (or “coarse-to-fine”) strategy, which consists of first computing the 1-layer model, copying the solution on two layers, and computing the 2-layer model, copying the solution on four layers, etc., until the prescribed number of layers is reached. This strategy can be combined to both extension methods presented in Sections 3.1 and 3.2.

4 Numerical results

In this section, one uses the MSSA model in a flow-line setting (i.e. $d = 2$) to carry out two types of modelling experiments. First, one tests the numerical performances of the methods presented in Section 3 with ISMIP-HOM experiments B and D [22]. Second, one tests the mechanical performances of the MSSA model with a MISMIP experiment [21]. For all runs presented here, the following physical parameters were used: $\rho = 910$ kg m$^{-3}$, $\rho_w = 1000$ kg m$^{-3}$, $n = 3$, and $g = 9.81$ m s$^{-2}$.

4.1 Numerical performance tests for ISMIP-HOM experiments B and D

ISMIP-HOM [22] experiments consist of modelling exercises based on various ice geometries and boundary conditions in order to generate different types of ice flows, which can be met in real glacier modelling. Experiments B, D and E were conducted in [13] for model comparison purposes. As a result, the MSSA solutions have shown good agreements with the higher-order ones in all experiments, see Figure 3. In contrast, the SIA and SSA solutions suit only specific ice flow and low aspect ratios. Here, the computational cost to reach a MSSA solution using the numerical methods of Section 3 is compared to the one needed to compute the First-Order Approximation (FOA) [1, 20] solution with a comparable method. For this purpose, one runs ISMIP-HOM experiments B and D since they represent a wide range of various ice flows (from shearing to sliding-dominant flows). Unlike [13] which considers also MSSA solutions based on second-order reconstructed interlayer tractions, here, only the simplest MSSA solution based on the zeroth-order reconstruction (22) and an uniform multilayer splitting is considered. Note that such a more complex formulation based on an a priori estimate of the streamlines as described [13] (or A) might improve the mechanical performance of the model in some cases. However, upgrading to such a formulation (i.e. from (22) to (40)) brings no further computational complexity. Numerical experiments (not shown) confirmed that those upgraded MSSA solutions are not more expensive to compute.

The goal of ISMIP-HOM experiments B and D is to compute diagnostic velocity fields for given domains of ice and boundary conditions. Since those ones show no lateral (in $y$) variation, a flow-line model is used to perform this task. More precisely, in experiment B, the geometry is defined by

$$\bar{s}(x) = -x \tan(0.5^\circ), \quad x \in \Omega = [0, X]$$

$$\bar{s}(x) = \bar{s}(x) - 1000 + 500 \sin \left(2\pi x / X \right), \quad x \in \Omega = [0, X]$$

and a no-slip condition is prescribed on the bedrock, while, in experiment D, the geometry
Figure 3: MSSA, FOA, SSA and SIA surface horizontal velocities for ISMIP-HOM experiment B (left) and D (right) with $X = 10$, 40 and 160 km (from the top to the bottom) from [13].
is defined by

\[ \tau(x) = -x \tan(0.1^\circ), \quad x \in \Omega = [0, X] \]
\[ s(x) = \tau(x) - 1000, \quad x \in \Omega = [0, X] \]

and a slip condition is prescribed everywhere on the bedrock defined by \( m = 1 \) and

\[ C(x) = 1000(1 + \sin(2\pi x/X)), \]

where \( X > 0 \) is given. In both experiments, periodic boundary conditions connect the left- and right-hand sides of \( \Omega \), and a stress-free condition is prescribed on the top surface. In addition, the following physical parameter \( A = 3.17 \times 10^{-24} \text{ Pa}^{-3} \text{ s}^{-1} \), see [22] for further details.

In both experiments, the horizontal segment \( \Omega \) was divided into 64 equal sized segments to generate a 1D uniform mesh. The resulting mesh and a uniform splitting of the ice thickness (defined by \( h^l = h/L \)) with 16 layers were used to compute the MSSA solutions. The accuracy of all numerical solutions was assessed by examining the discrepancy between these solutions and those obtained by doubling the horizontal resolution and the number of layers. For further confidence, the gradient and the viscosity fields of the MSSA solutions were also checked. As result, all fields were always found convergent when refining the horizontal mesh, the multilayer vertical splitting or both simultaneously.

As seen in Section 3, any numerical model can be characterised by

a) the extension method: layer-wise or column-wise (Sections 3.1 and 3.2),

b) the initialisation by \( u = 0 \) or by the nested strategy, see Section 3.3.

For the sake of convenience, the methods are labelled as follows:

EXTENSION-INITIALISATION,

where ‘EXTENSION’ is replaced by ‘L’ for Layer-wise (Section 3.1) or ‘C’ for Column-wise (Section 3.2), and ‘INITIALISATION’ is replaced by ‘0’ for an initialisation by \( u = 0 \) or by ‘N’ for a Nested initialisation (Section 3.3). For example, the C-N method means the Column-wise extension with the Nested strategy. To analyse which features of the method improve the performance of the global solver, the number of “outer” iterations to reach convergence was recorded for different methods. Here, one “outer” iteration applies by definition to all nodes and all layers (\( L \times I \times 1 \) as defined in Section 3). Table 1 displays the number of outer iterations needed to obtain the convergence of the 1-layer, 2-layer, 4-layer, 8-layer and 16-layer MSSA solutions with both types of extensions of the Newton multigrid solver [14] and with the two initialisation strategies. More precisely, the two upper blocks in Table 1 display the results obtained by computing each \( L \)-layer solution independently starting from \( u = 0 \). In contrast, the two lower blocks of Table 1 display the results obtained by computing the \( L \)-layer solution sequentially starting from the \( L/2 \)-layer solution to compute the \( L \)-layer one (following the nested initialisation described in Section 3.3). In practice, the overall CPU time is broadly proportional to the number of iterations and the number of horizontal nodes, for a given number of layers. However, using the nested strategy requires \( L \)-layer systems to be solved with various \( L \). As a consequence, the only way to compare the efficiency of each method is to compare the CPU times needed to compute the final 16-layer solutions at a given accuracy. Table 2 displays these CPU times for various extension methods and initialisation strategies. All
Table 1: Number of outer iterations to reach the convergence of MSSA solutions of ISMIP-HOM experiments B and D using various extension methods and initialisation strategies. When \( L = 1 \), all methods coincide such that the result is displayed only once in the first line. For convenience, the aspect ratio \( \epsilon \) of the geometry is indicated for each experience.

<table>
<thead>
<tr>
<th>Method</th>
<th>L</th>
<th>( X = 10 )</th>
<th>( X = 40 )</th>
<th>( X = 160 )</th>
<th>( \epsilon = 0.1 )</th>
<th>( \epsilon = 0.025 )</th>
<th>( \epsilon \sim 0.006 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>1</td>
<td>485</td>
<td>125</td>
<td>78</td>
<td>169</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>L-0</td>
<td>2</td>
<td>709</td>
<td>252</td>
<td>103</td>
<td>&gt;999</td>
<td>&gt;999</td>
<td>&gt;999</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>&gt;999</td>
<td>512</td>
<td>240</td>
<td>&gt;999</td>
<td>&gt;999</td>
<td>&gt;999</td>
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<tr>
<td></td>
<td>8</td>
<td>&gt;999</td>
<td>&gt;999</td>
<td>&gt;999</td>
<td>&gt;999</td>
<td>&gt;999</td>
<td>&gt;999</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>&gt;999</td>
<td>&gt;999</td>
<td>&gt;999</td>
<td>&gt;999</td>
<td>&gt;999</td>
<td>&gt;999</td>
</tr>
<tr>
<td>C-0</td>
<td>2</td>
<td>361</td>
<td>107</td>
<td>56</td>
<td>456</td>
<td>17</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>273</td>
<td>75</td>
<td>31</td>
<td>388</td>
<td>37</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>226</td>
<td>86</td>
<td>61</td>
<td>213</td>
<td>44</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>133</td>
<td>78</td>
<td>56</td>
<td>219</td>
<td>44</td>
<td>24</td>
</tr>
<tr>
<td>L-N</td>
<td>2</td>
<td>943</td>
<td>335</td>
<td>78</td>
<td>396</td>
<td>20</td>
<td>2</td>
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<td></td>
<td>4</td>
<td>899</td>
<td>311</td>
<td>130</td>
<td>409</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>180</td>
<td>152</td>
<td>154</td>
<td>328</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>37</td>
<td>13</td>
<td>12</td>
<td>137</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>C-N</td>
<td>2</td>
<td>348</td>
<td>266</td>
<td>75</td>
<td>462</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>221</td>
<td>165</td>
<td>74</td>
<td>353</td>
<td>15</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>146</td>
<td>63</td>
<td>54</td>
<td>159</td>
<td>34</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>100</td>
<td>39</td>
<td>38</td>
<td>102</td>
<td>8</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 2: CPU times (in seconds) needed to compute the MSSA solutions of ISMIP-HOM experiments B and D using various extension methods and initialisation strategies. For the methods coupled to the Nested strategy, the cumulative time is displayed.

<table>
<thead>
<tr>
<th>Method</th>
<th>L</th>
<th>( X = 10 )</th>
<th>( X = 40 )</th>
<th>( X = 160 )</th>
<th>( \epsilon = 0.1 )</th>
<th>( \epsilon = 0.025 )</th>
<th>( \epsilon \sim 0.006 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>1</td>
<td>0.99</td>
<td>0.22</td>
<td>0.16</td>
<td>0.61</td>
<td>0.03</td>
<td>0.02</td>
</tr>
<tr>
<td>L-0</td>
<td>16</td>
<td>&gt; 99</td>
<td>&gt; 99</td>
<td>&gt; 99</td>
<td>&gt; 99</td>
<td>&gt; 99</td>
<td>&gt; 99</td>
</tr>
<tr>
<td>C-0</td>
<td>16</td>
<td>5.53</td>
<td>3.39</td>
<td>3.89</td>
<td>18</td>
<td>3.73</td>
<td>2.22</td>
</tr>
<tr>
<td>L-N</td>
<td>16</td>
<td>38.9</td>
<td>19.8</td>
<td>13.7</td>
<td>67.9</td>
<td>1.97</td>
<td>0.86</td>
</tr>
<tr>
<td>C-N</td>
<td>16</td>
<td>11.6</td>
<td>7.22</td>
<td>4.14</td>
<td>24.4</td>
<td>2.24</td>
<td>0.54</td>
</tr>
</tbody>
</table>

The first block of Table 1 indicates that the convergence of the layer-wise extension L-0 method shrinks dramatically when increasing the number of layers such that this method is not usable in practical applications. As explained in Section 3.1, increasing the number of layers renders the transfer of information between layers more expensive, in particular between the lowest and the highest layer. In contrast, the column-wise extension C-0 method proves to be much more robust since the number of iterations does not increase with the number of layers, as indicated in the second block of Table 2. It
is remarkable to note that the nested (or “coarse-to-fine”) initialisation strategy (Section 3.3), which facilitates the vertical transfer of information, recovers the convergence of the layer-wise extension L-N method, see the third block of Table 1. As expected, the improved initialisation also decreases the number iterations needed to reach convergence of the column-wise extension C-N method, however, to a lesser extent. For all methods Table 1 also shows that fewer iterations are needed to compute the flows for large wavelengths $X$ of the domain, or equivalently for small aspect ratios. Indeed decreasing the aspect ratio lowers the contribution of the p-Laplace terms, in which derivatives in $x$ (scaled by $X$) are involved, in the MSSA system. In contrast, the interlayer and basal terms are not affected. As a consequence, large wavelengths $X$ render the matrices of the MSSA system sparser and the solver faster.

In terms of CPU time, Table 2 clearly confirms that the layer-wise method (L-0) is not usable unless it is combined to the nested initialisation strategy (L-N). However, Table 2 indicates that this combination (L-N) is competitive (compared to other methods) only for the sliding-dominant ice flows of experiments D with $X = 40$ and 160. In contrast, the column-wise extension C-0 method shows the best performances for almost all experiments, except for experiment D with the same large wavelengths $X$, for which the nested initialisation C-N method further reduces the computational time. Indeed, in these cases, the solution is nearly vertically constant such that a single layer is enough to get very close to the solution.

To complete this study, the cost to compute the MSSA solution (obtained with the C-0 method since it was found the most efficient in most of cases) and the FOA one are compared. To obtain the FOA solution which can be fairly compared to the MSSA one, a triangular 2D mesh was built by extruding uniformly 16 vertical layers of the 1D mesh between the lower and the upper surfaces, and splitting each rectangle into two triangles. Thus, the FOA solution is computed by finite elements using continuous piecewise linear functions on the 2D mesh. A Newton method was used to tackle the FOA non-linearity, however, with a linear Gauß–Seidel solver instead of a multigrid one to solve the correction step. Indeed, it is complex to build a 2D mesh of a non regular shape which owns a hierarchical structure as it is necessary to apply the geometrical multigrid V-cycles. In contrast, doing so is straightforward for a segment by successive intervals splitting. In order to lead a fair comparison of the computational costs associated to the MSSA and the FOA solutions, the linear solvers for Newton correction step must be similar. For this reason, the MSSA solution is recomputed using a linear Gauß–Seidel solver instead of a multigrid one. This method is abbreviated C-0-GS to distinguish it from the C-0 method which was used until now. Table 3 and Table 4 display the number of outer iterations and the CPU times needed to obtain the convergence of the MSSA and the FOA solutions.

<table>
<thead>
<tr>
<th>Model</th>
<th>Method</th>
<th>Experiment B</th>
<th>Experiment D</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$X = 10$</td>
<td>$X = 40$</td>
</tr>
<tr>
<td>MSSA</td>
<td>C-0</td>
<td>133</td>
<td>78</td>
</tr>
<tr>
<td>MSSA</td>
<td>C-0-GS</td>
<td>255</td>
<td>76</td>
</tr>
<tr>
<td>FOA</td>
<td>GS</td>
<td>46</td>
<td>34</td>
</tr>
</tbody>
</table>

Table 3: Number of outer iterations to reach the convergence of the solutions of ISMIP-HOM experiments B and D using various models and methods.

As expected, using the multigrid solver instead of the Gauß–Seidel one to achieve the Newton correction improves the performances of the overall method, however, substantial
cost reductions are only visible at low wavelengths \( X \) of both experiments B and D. This can be explained as follows: Since \( x \) is scaled by \( X \), horizontal derivatives get negligible compared to vertical ones when \( X \) is large such that the coupling between the columns gets weaker. With a little coupling, the sequential Gauß–Seidel is expected to converge rapidly. In contrast, the multigrid solver shows its full efficiency when the horizontal coupling is strong as at low wavelengths \( X \).

When using a similar numerical method (Newton correction achieved by a linear Gauß–Seidel method), the FOA solution is found cheaper to compute than the MSSA one in the case of experiment B, while, this is the opposite in the case of experiment D. This can be justified by the type of flow. In the case of experiment B, the vertical gradients are dominant (furthermore at low wavelengths \( X \)). In contrast, the solution varies less with \( z \) when introducing some sliding in experiment D (furthermore at high wavelengths \( X \)). The consequence of this anisotropy is that it severely increases the condition number of the matrix \([2]\), and subsequently deteriorates the convergence of the linear Gauß–Seidel solver. Note that this convergence can be improved by implementing a preconditioner that accounts for the problem anisotropy, like the multigrid one proposed in \([2]\). However, this later requires a grid hierarchy which is not only horizontal, but also vertical, making its implementation further complex.

From \([13]\) and the results of this section, one can assess the computational and mechanical strengths and weaknesses of the MSSA solutions computed using the column-wise extension method relatively to the more general FOA solutions for different types of flow. When computing the shearing-dominant flows of experiment B, the FOA solution proved to be computationally cheaper, while being mechanically further accurate, in particular, at low wavelengths \( X \), see Figure 3 (left). For this type of flows, the only advantage of the MSSA solution is its simplicity of implementation since it requires to mesh only the horizontal domain. In contrast, when computing the sliding-dominant flows of experiment D, the MSSA solutions were found much cheaper to compute than the FOA’s ones while being very close to FOA solutions, see Figure 3 (right). As a consequence, the multilayer formulation seems to be naturally better conditioned than the FOA one to face anisotropic conditions. This can be justified by the differences of discretisations between the horizontal and vertical directions. In conclusion, the MSSA model is expected to be well-suited to compute the flows in marine sheets since there are similar to those of experiment D.

### 4.2 Mechanical performance test for a marine ice sheet experiment

In this section, one runs MISMIP experiment 1a/2a \([21]\) applying the MSSA model with 1, 2, 4, 8 and 16 layers for comparison purposes. Similar results were presented in \([14]\), however, only for the SSA and the SIA+SSA models \([3]\). For convenience, the settings are briefly recalled, see \([21]\) for further details. Let \( \Omega = [0, 2000] \) km be a one-dimensional

<table>
<thead>
<tr>
<th>Model</th>
<th>Method</th>
<th>Experiment B</th>
<th>Experiment D</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( X =10 )</td>
<td>( X =40 )</td>
</tr>
<tr>
<td>MSSA</td>
<td>C-0</td>
<td>5.53</td>
<td>3.39</td>
</tr>
<tr>
<td>MSSA</td>
<td>C-0-GS</td>
<td>15</td>
<td>4.66</td>
</tr>
<tr>
<td>FOA</td>
<td>GS</td>
<td>2.45</td>
<td>2.05</td>
</tr>
</tbody>
</table>

Table 4: CPU times (in seconds) needed to compute the solutions of ISMIP-HOM experiments B and D using various models and methods.
domain, and

\[ b(x) = 720 - 778.5 \left( x/750000 \right), \]

a function describing the bedrock. The following boundary conditions are prescribed: the zero-velocity condition \( \mathbf{u} = 0 \) is imposed at \( x = 0 \) km, the sliding law (4) is prescribed under the bedrock, and the condition (19) is imposed at the calving front which is always at \( x = 2000 \) km. The sliding law applies only under the grounded area \( [0, x_{GL}] \) with the parameters \( C = 7.624 \times 10^6 \text{ Pa m}^{-m} \text{s}^m \) and \( m = 1/3 \), while \( C = 0 \) on the floating area \( [x_{GL}, 2000] \), where \( x_{GL} \) is the abscissa of the GL. Finally, the accumulation rate in (28) is taken as constant, \( a = 0.3 \text{ m a}^{-1} \).

As in [14], the domain \( \Omega \) is first uniformly meshed with \( \sim 14 \) km-long-element, and second adaptively remeshed to strongly refine close to the GL with \( \sim 0.2 \) km-long-element and capture accurately the high gradients expected in this area. On the one hand, the column-wise extension of the Newton Multigrid method [14] with the nested initialisation strategy (see Section 3) is used to compute the MSSA solution, since this method has shown the best performances in Section 4.1. On the other hand, the mass conservation equation (28) is solved by using a vertex-centred finite volume method [14]. Note that the motion of the GL is driven implicitly by the equation (28), which redefines at each time step the new ice thickness \( h \) and the new position the GL via the flotation criterion (2). In general, the GL does not match any node of the discretisation. To prevent against artefacts caused by a numerical approximation of the GL, the sliding term (15) is integrated over the grounded part \( [0, x_{GL}] \) exactly.

For a given rate factor \( A \), and starting from an empty ice geometry \( (h = 0 \text{ on } \Omega) \), the given configuration allows an ice sheet to grow with time, with a grounded part on the left-hand-side of \( \Omega \) and a floating part on the right-hand-side of \( \Omega \), and to stabilize when the incoming surface mass balance and the outgoing ice flow at the calving front \( x = 2000 \) km compensate. In MISMIP experiments, the model is run sequentially with 9 decreasing rate factors \( A \) which control the ice softness: from \( A = 4.6416 \times 10^{-24} \text{ to } A = 1 \times 10^{-26} \text{ Pa}^{-3} \text{ s}^{-1} \). For each \( A \), one records the GL position once the steady shape is reached, before to switch to the next parameter \( A \) (the ice sheet is considered to be steady if the normalized annual volume change is lower than \( 10^{-6} \)). By decreasing \( A \), the velocity and the flux at the GL are reduced too, such that the GL must reach a steady position further from the origin. Once the last parameter value is reached, the same procedure is applied, however, looping backward over the parameter values \( A \), i.e. from the lowest one to the highest one. This additional experiment aims to test the reversibility of the GL, i.e. its ability to recover its position after being perturbed by changes in parameter value. For all \( A \), the experiment provides one position of the advancing GL and one for the retreating GL, see Figures 4 and 5.

Before analysing the results of Figures 4 and 5, it is instructive to first compare the stationary MSSA velocity fields obtained with 1, 2, 4, 8 and 16 layers. Figure 6 displays the vertically averaged velocities of all models computed with the highest rate factor \( A = 4.6416 \times 10^{-24} \text{ Pa}^{-3} \text{ s}^{-1} \) and the first steady geometry obtained with the 1-layer model of MISMIP experiment 1a/2a. As a matter of fact, the magnitude of the MSSA solution increases and converges when increasing the number of layers. This illustrates how using more than one layer softens the ice by the contribution of the vertical shear stresses, resulting in increased velocities and fluxes. From a computational point of view, it took \( \sim 30 \) and 375 seconds to reach the convergence of the 16-layer MSSA solution of Figure 5 starting for zero with the C-0 and the C-0-GS (replacing the linear multigrid solver by a linear Gauß-Seidel one) methods, respectively. Here using a multigrid linear solver instead of the Gauß-Seidel one improves dramatically the convergence and the overall
Figure 4: Parameter $A$ with respect to the GL position (top) and the height of ice at $x = 0$ (bottom) for experiment 1a/2a obtained with the 1-layer, 4-layer and 16-layer MSSA solutions (from the left to the right). The analytic solution from [27] is drawn with the bold line.

Figure 5: GL position and height of ice at $x = 0$ corresponding to the first steady state reached in experiments 1a/2a, i.e. advancing stage with $A = 4.6416 \times 10^{-24}$ Pa$^{-3}$ s$^{-1}$.

As in [14] for the SSA, the adaptive mesh with a strong refinement close to the GL ensures the GL to be nearly reversible after re-increasing the rate factor, see Figure 6. A closer look at Figure 4 (top) indicates that the reversibility is slightly harmed when increasing the number of layers to compute the MSSA solution. More precisely, during the
advancing stage, the GL tends to stabilize slightly closer to the origin when using further layers. This phenomena is further visible for the highest values of $A$ which correspond to the fastest flows: Figure 5 indicates a discrepancy of about 15 km between the GL modelled with the SSA and the one modelled with the 16-layer MSSA. Interestingly, during the retreating stage, all MSSA solutions follow accurately the same set of positions, which correspond to those given by the semi-analytic solution proposed in [27] or equivalently the ones of the SSA model. The difference between the advancing and retreating GL positions can be seen as a measure of numerical error. Indeed, an additional experiment (not shown) with a greater accuracy close to the GL (0.1 km instead of 0.2 km) shows a reduction of the discrepancy (from 15 km to 12 km). In conclusion, adding layers in the MSSA does not change significantly the positions of the GL in a steady regime, but requires to refine further the mesh to preserve the reversibility of the GL. Figures 4 (bottom) and 5 also display an interesting feature of the MSSA model: The global thickness of the grounded ice decreases when increasing the number of layers, and this thinning is even more pronounced for the highest rate factors $A$. This is the consequence of higher velocities induced by the the interlayer vertical shear stresses, especially on the grounded part, as shown in Figure 6. In contrast, it is worth noting that the SSA underestimates the ice velocities and overestimates the ice thickness.

5 Conclusions and perspectives

In this study, the MSSA ice flow model [13] was considered. Through the construction of the MSSA, the ice thickness consists of a pile of thin layers which can spread out, contract and slide over each other. Unlike the SSA, the velocity profile of the MSSA solution is vertically piecewise-constant on each layer and the vertical shear stresses generated by interlayer sliding are accounted for. As a result, the MSSA is a multilayer generalisation of the SSA and forms a tridiagonal system of non-linear elliptic equations similar to the SSA. In this paper, the variational and the minimisation forms of the MSSA system were derived. Interestingly, the functional to be minimised gives a new interpretation of the model: Compared to the SSA one, the MSSA functional involves interlayer sliding terms.
which can be seen as penalisation terms for the jumps of velocity across the layers, the norm of the penalisation being defined naturally by Glen’s law parameters.

Taking advantage of the minimisation formulation, two numerical methods were presented to solve the MSSA system as an extension of an existing solver for the SSA, like the multigrid Newton one described in [14]. The first one (the layer-wise extension) involves a simple iterative method to loop over layers. Numerical results for prognostic flow-line ISMIP-HOM experiments have shown that this method is efficient for computing sliding-dominant ice flows like those of ice shelves, but only if it is combined to a nested initialisation strategy. However, its convergence rate severely deteriorates when computing other types of flows. To overcome this problem, the second method (the column-wise extension), which reverses the order of loops (horizontal over nodes or vertical over layers), was found much more robust. In particular, its convergence time proved to be relatively insensitive to the type of ice flows.

When comparing the computational performances of both formulations (FOA and MSSA) with a similar method, the MSSA one proved to be naturally better conditioned than the FOA one to face highly anisotropic conditions as in ISMIP-HOM experiment D which involves sliding-dominant ice flows. In contrast, flows dominated by vertical shear remain more efficiently computed with a standard FOA formulation while being mechanically further accurate. When running the MISMIP 2D 1a/2a experiment, no substantial change in the behaviour of the GL was found between the SSA and the MSSA models. However, in comparison with the SSA, the MSSA model yields to more realistic velocity fields and ice thickness over the grounded area thanks to the presence of the vertical shear stresses in the MSSA formulation.

As a follow-up of this paper, further model comparisons (mechanical and numerical) should be realised for prognostic 3D simulations of ice sheets [32] or mountain glaciers [15] to better evaluate the capabilities of the MSSA in real cases of modelling. In this last application, glacier surface slopes might be sufficiently high such that the second-order MSSA model described in [14] or A might be valuable to improve the solution of the MSSA model considered in this paper. An other aspect to investigate is to use the MSSA for preconditioning the FOA or Stokes problem regarding the multilayer approach as a way of simplifying the final matrix by setting small coupling terms to zero. Finally, the ice was assumed to be isothermal in this paper for simplicity. However, this assumption does not apply to many situations of modelling (like the ice sheets). Consequently, it will be necessary to investigate in a future study how the conservation of energy can be included to the MSSA model, and how to discretize it in compliance with the multilayer approach.

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References


A Multilayer model with second-order accurate interlayer tractions

In [13], an improved multilayer model which exploits an a priori knowledge of the streamlines were proposed. In this model, the redefinition of $S^l$ and $-S^{l-1}$ was further complex. To derived second-order one, two assumptions are done, see the details in [13]. First, the layers are assumed to be aligned with the ice flow (however, since the direction of the flows can not be known without solving the Stokes model, the vertical division (11) must be chosen such that the layers are approximatively oriented with the ice flow). Second the layers are assumed to be sufficiently thin such that the $O(\epsilon \delta / L)$ terms can be neglected, where $\epsilon$ is the aspect ratio and $\delta$ is the magnitude of the interlayer slopes. In contrast, terms in $O(\delta^2)$ are conserved. Keeping the only vertical shear stress components in the local frame of the interface, the interlayer tractions are redefined by

$$S^l = A^{-\frac{1}{\alpha}} \left( \alpha' \right)^{\frac{1}{2} + 1} \left| \frac{u^{l+1} - u^l}{h^{l+1} + h^l} \right| M^l \left( \frac{u^{l+1} - u^l}{h^{l+1} + h^l} \right), \quad l \in \{1, ..., L\},$$  

(31)

where

$$\alpha^l := \sqrt{1 + |\nabla s|^2},$$  

(32)

and the norm $|\cdot|_{M^l}$ derives from the scalar product

$$|u^2|_{M^l} = u^T M^l u,$$  

(33)

where the $2 \times 2$ matrix $M^l$ is defined by

$$M^l = I + (\nabla s^l)(\nabla s^l)^T.$$  

(34)

In the same way, the conditions on the bedrock (15) rewrites:

$$S^0 = A^{-\frac{1}{\alpha}} \left( \alpha^0 \right)^{\frac{1}{2} + 1} \left| \frac{u^1}{h^1} \right|_{M^0} \left( M^0 \left( \frac{u^1}{h^1} \right) \right) \times 1_{\Omega_0}$$  

(35)

$$+ C \alpha^0 \left| \frac{u^1}{h^1} \right|_{M^0} \left( M^0 u^1 \right) \times 1_{\Omega_m}. \quad (36)$$

Interestingly, (36) reduces to (4) when neglecting $O(\delta^2)$ terms. It follows that the MSSA solution $(u^1, ..., u^L)$ solves the following $2 \times 2$-block tridiagonal system of two-dimensional non-linear elliptic equations [13]:

$$- A^{-\frac{1}{2}} \nabla \cdot \left( h^L |D(u^L)|^{\frac{1}{2} - 1} \left( D(u^L) + \text{tr}(D(u^L)) I \right) \right)$$  

(37)

$$+ A^{-\frac{1}{2}} \left( \alpha^{L-1} \right)^{\frac{1}{2} + 1} \left| \frac{u^L - u^{L-1}}{h^{L-1} + h^L} \right|_{M^{L-1}} \left( M^{L-1} \left( \frac{u^L - u^{L-1}}{h^{L-1} + h^L} \right) \right) = - \rho g h^L \nabla s,$$
for all \( l \in \{2, \ldots, L-1\} \):

\[
-A^{-\frac{1}{\alpha}} \nabla \cdot \left( h^{1} \left| D(u^{l}) \right|^{\frac{1}{\alpha}-1} [D(u^{l}) + \text{tr}(D(u^{l}))I] \right) + A^{-\frac{1}{\alpha}} \left( \alpha^{l-1} \right) \frac{1}{\alpha} \left| \frac{u^{l}-u^{l-1}}{h^{l-1}+h^{l}} \right|_{M^{-1}}^{\frac{1}{\alpha}-1} \left[ M^{l-1} \left( \frac{u^{l}-u^{l-1}}{h^{l-1}+h^{l}} \right) \right] = -\rho gh^{l} \nabla \varphi, \tag{38}
\]

and

\[
-A^{-\frac{1}{\alpha}} \nabla \cdot \left( h^{1} \left| D(u^{l}) \right|^{\frac{1}{\alpha}-1} [D(u^{l}) + \text{tr}(D(u^{l}))I] \right) + A^{-\frac{1}{\alpha}} \left( \alpha^{l-1} \right) \frac{1}{\alpha} \left| \frac{u^{l}-u^{l-1}}{h^{l-1}+h^{l}} \right|_{M^{-1}}^{\frac{1}{\alpha}-1} \left[ M^{l-1} \left( \frac{u^{l}-u^{l-1}}{h^{l-1}+h^{l}} \right) \right] = \rho gh^{l} \nabla \varphi. \tag{39}
\]

At the boundary of the ice domain \( \partial \Omega \), the condition (19) still prevails. The variational formulation associated to (37) (38) (39) and (19) writes

\[
A^{-\frac{1}{\alpha}} \sum_{l=1}^{L} \int_{\Omega} h^{l} |D(u^{l})|^{\frac{1}{\alpha}-1} (D(u^{l}), D(v^{l})) d\Omega \tag{40}
\]

\[
+ A^{-\frac{1}{\alpha}} \int_{\Omega_{0}} \left( \alpha^{l-1} \right) \frac{1}{\alpha} \left| \frac{u^{l}-u^{l-1}}{h^{l-1}+h^{l}} \right|_{M^{-1}}^{\frac{1}{\alpha}-1} \left[ M^{l-1} \left( \frac{u^{l}-u^{l-1}}{h^{l-1}+h^{l}} \right) \right] \cdot v^{l} d\Omega \tag{42}
\]

\[
+ \int_{\Omega_{m}} \alpha^{l} \left| u^{l} \right|_{M^{0}}^{\frac{1}{\alpha}} (M^{0} u^{l}) \cdot v^{l} d\Omega \tag{43}
\]

\[
+ A^{-\frac{1}{\alpha}} \sum_{l=2}^{L} \int_{\Omega} \left( \alpha^{l-1} \right) \frac{1}{\alpha} \left| \frac{u^{l}-u^{l-1}}{h^{l-1}+h^{l}} \right|_{M^{-1}}^{\frac{1}{\alpha}-1} \left[ M^{l-1} \left( \frac{u^{l}-u^{l-1}}{h^{l-1}+h^{l}} \right) \right] \cdot (v^{l} - v^{l-1}) d\Omega \tag{44}
\]

Finally, like in Section 2.2, one can verify that (40) is the Euler-Lagrange equation, \( \langle DJ(u), v \rangle = 0 \), where \( u = (u^{1}, \ldots, u^{L}) \) and \( v = (v^{1}, \ldots, v^{L}) \), for the functional

\[
J(u) = \frac{A^{-\frac{1}{\alpha}}}{\alpha} \sum_{l=1}^{L} \int_{\Omega} h^{l} |D(u^{l})|^{\frac{1}{\alpha}+1} d\Omega \tag{41}
\]

\[
+ \frac{A^{-\frac{1}{\alpha}}}{\alpha} \int_{\Omega_{0}} \alpha^{0} \left( \frac{1}{h^{1}} \right) \frac{1}{\alpha} \left| u^{1} \right|_{M^{0}}^{\frac{1}{\alpha}+1} + \frac{1}{m+1} \int_{\Omega_{m}} \alpha^{0} \left| u^{1} \right|_{M^{0}}^{\frac{1}{\alpha}+1} d\Omega \tag{42}
\]

\[
+ \frac{A^{-\frac{1}{\alpha}}}{\alpha} \sum_{l=2}^{L} \int_{\Omega} \left( \alpha^{l-1} \right) \frac{1}{\alpha} \left| \frac{u^{l}-u^{l-1}}{h^{l-1}+h^{l}} \right|_{M^{-1}}^{\frac{1}{\alpha}+1} d\Omega + \rho g \int_{\Omega} h^{l} \nabla \varphi \cdot u^{l} d\Omega - \int_{\Omega} F^{l} n \cdot u^{l} dS. \tag{44}
\]