An Internal Stability Example

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To illustrate the concept of internal stability we will look at an example where there are several pole-zero cancellations between the controller and the plant. One of them, on the $j\omega$-axis, will be an exact cancellation; and the other, in the left-half plane, will be an inexact cancellation.

The control design illustrated here is a very bad idea and it leads to an unstable closed-loop system. The fundamental observation is that $j\omega$-axis and right-half plane pole-zero cancellations will lead to an unstable transfer function in the set of transfer functions to be checked when testing internal stability.

1 Plant and controller

The plant is,

$$G(s) = \frac{1}{s(1-s/a)}, \qquad \text{where} \qquad a = -2.$$

This plant has an integrator and a pole at $s = a$.

The controller is given by,

$$K(s) = \frac{200s(1-s/a_K)}{(1+s/0.1)(1+s/50)^2}.$$

The controller as zeros at $s = 0$ and $s = a_K$. The a_K zero can be considered as an attempt to cancel the plant pole at $s = a$. For this example we take,

$$a_K = -2.1,$$

and so the cancellation is not exact.

*modified: 26 April 2017: signal label for controller output changed from u to z to match other figures in the lectures.

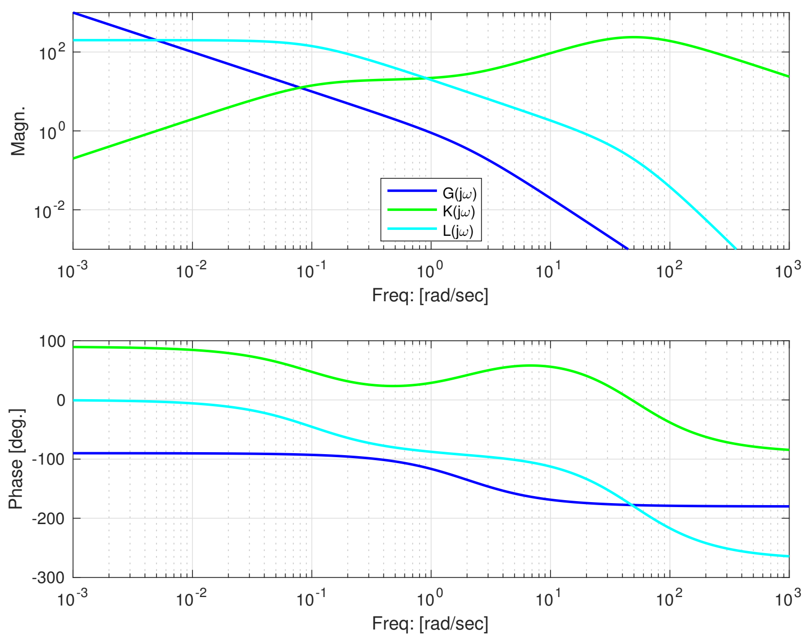
2 Loopshape analysis

The loopshape, $L(s) = G(s)K(s)$, is

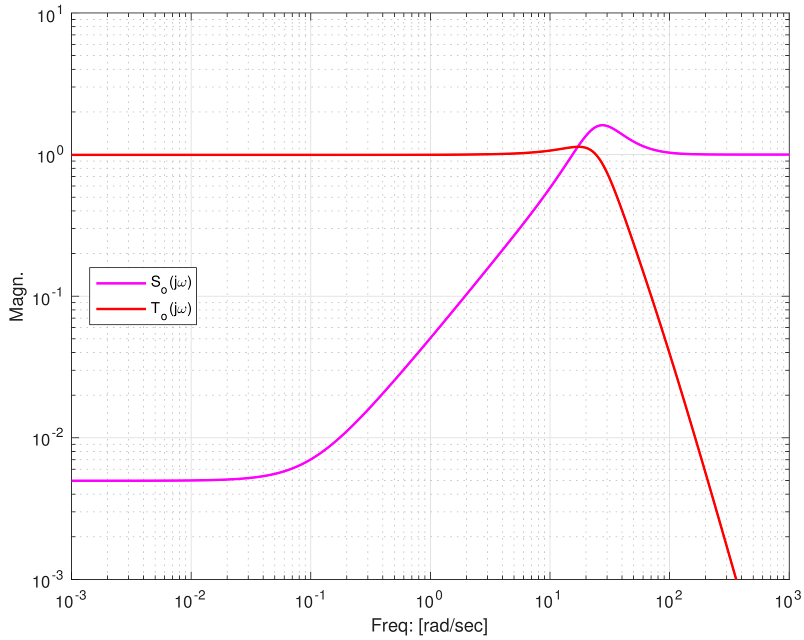
$$L(s) = \frac{(1 - s/a_K)}{(1 - s/a)} \frac{200}{(1 + s/0.1)(1 + s/50)^2}.$$

The inexact pole-zero cancellation at $s = a$ changes the maximum frequency domain response of the loopshape by approximately 5%. This is not a significant problem as the pole in this term (at $s = a$) is stable.

The Bode plot of the loopshape, suggests that the closed-loop system will be stable and have reasonable gain and phase margins. Note that the exactly cancelled pole at $s = 0$ will not enter into the loopshape analysis.



This gives sensitivity and complementary sensitivity functions, $S(s)$ and $T(s)$ respectively, that are reasonable. Note that both would be better if we didn't cancel the pole at $s = 0$.



If we calculate a minimal $S(s)$, or $T(s)$, and examine the closed-loop poles we find that they are at,

$$p_i = \{-75.2, -12.4 \pm j21.9, -2.11\}.$$

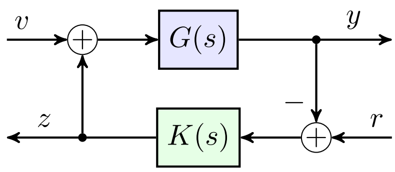
The transmission zeros of the minimal $S(s)$ are

$$z_i = \left\{-2, -10, -50 \pm j1.4 \times 10^{-6}\right\}.$$

Note that there are only four poles and all are stable. The pole that has been cancelled at $s = 0$ does not appear.

3 Internal stability analysis

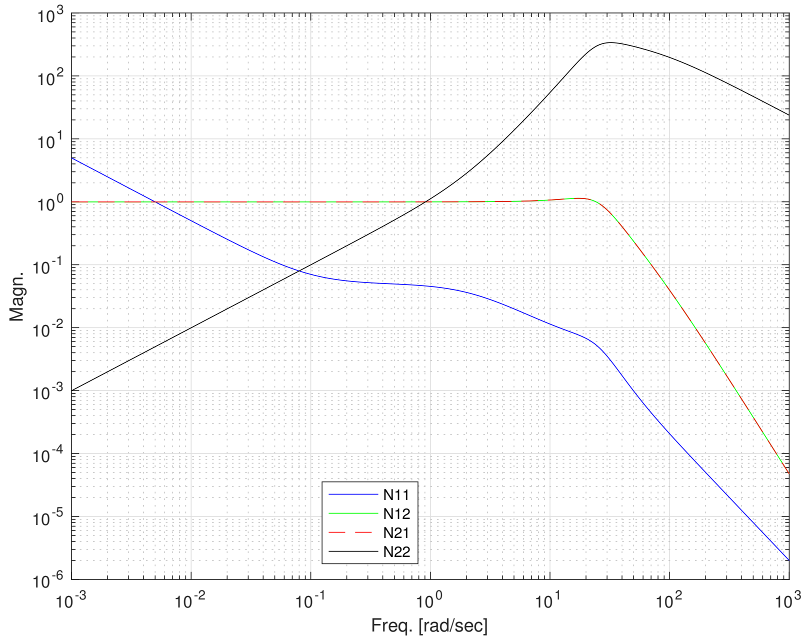
A more complete internal stability analysis looks at the four transfer functions in the interconnection shown below.



The input-output relationships shown are

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix} = \begin{bmatrix} S_{\mathrm{O}}(s)G(s) & T_{\mathrm{O}}(s) \\ -T_{\mathrm{I}}(s) & S_{\mathrm{I}}(s)K(s) \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}.$$

The Bode magnitude plots of each of these four transfer functions is shown. It is clear that the integrator appears on the $S_{\mathrm{O}}(s)G(s)$ transfer function.



We can calculate the two-input, two-output state-space realization for $N(s)$, and calculate it's poles at,

$$p_i = \{-75.2, -12.4 \pm j21.9, -2.11, 0\}.$$

If we look at the $N_{11}(s)$ transfer function and calculate its zeros they are

$$z_i = \{-50, -50, -10\}.$$

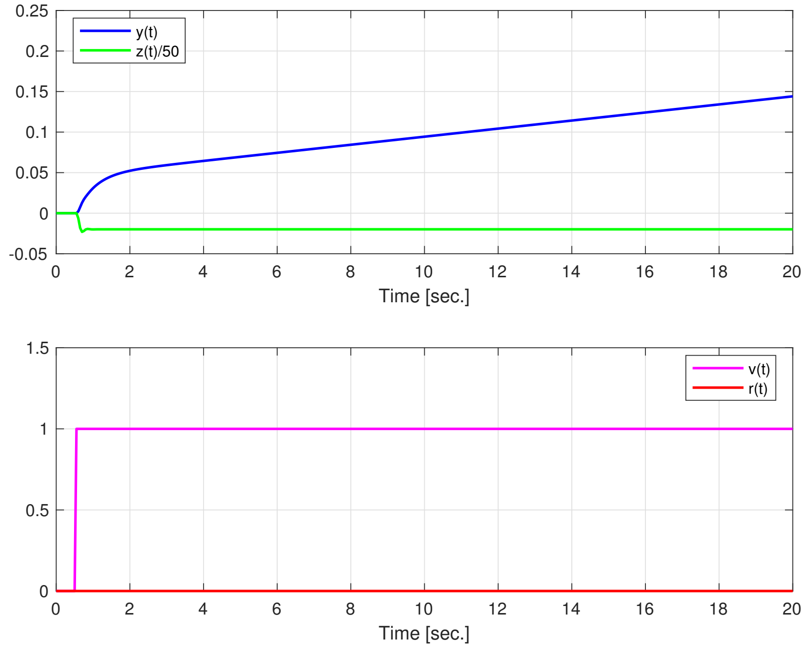
These zeros do not cancel any of the poles so all of the poles are evident in the $N_{11}(s)$ transfer function illustrated on the Bode plot[1].

To illustrate the instability we can simulate $N(s)$ with the input,

$$\begin{bmatrix} v(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \mathrm{step}(t-0.5) \\ 0 \end{bmatrix}, \quad \text{where step}(t) \text{ is the unit step function.}$$

All four signals are illustrated below. It is clear that $y(t)$ is unbounded.

[1]If we look at the transmission zeros for the entire $N(s)$ transfer function we see that it has zeros, $z_i = \{-2.1, 0\}$. The fact that $N(s)$ loses rank at $s = 0$ does not necessarily mean that all of the input-output transfer functions in $N(s)$ are zero and so the pole at $s = 0$ can appear in a component transfer function—in this case $N_{11}(s)$.



4 State-space viewpoint

Suppose we have minimal state-space realizations for the plant and controller in our interconnection. The equations corresponding to this are,

$$\begin{aligned} \frac{dx_G}{dt} &= A_G x_G + B_G(v+u) \\ y &= C_G x_G + D_G(v+u) \end{aligned}$$

and

$$\begin{aligned} \frac{dx_K}{dt} &= A_K x_K + B_K(r-y) \\ u &= C_K x_K + D_K(r-y). \end{aligned}$$

For simplicity in the following define,

$$E = (I - D_K D_G)^{-1}.$$

Note that if this inverse does not exist then the interconnection is not *well-posed*. In other words the equations actually have an infinite number of solutions. If either $D_K = 0$ or $D_G = 0$ then the inverse exists. As this is commonly the case we'll simply assume here that this inverse exists.

Somewhat involved algebra allows us to express the complete interconnection as a single state-space system.

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}x_G\\x_K\end{bmatrix} &= \begin{bmatrix}A_G+B_GED_KC_G & B_GEC_K\\ -B_KC_G-B_KD_GED_KC_G & A_K-B_KD_GEC_K\end{bmatrix}\begin{bmatrix}x_G\\x_K\end{bmatrix}\\ &\qquad + \begin{bmatrix}B_GE & -B_GED_K\\ -B_KD_GE & B_K+B_KD_GED_K\end{bmatrix}\begin{bmatrix}v\\r\end{bmatrix}\\ \begin{bmatrix}y\\u\end{bmatrix} &= \begin{bmatrix}C_G+D_GED_KC_G & D_GEC_K\\ ED_KC_G & EC_K\end{bmatrix}\begin{bmatrix}x_G\\x_K\end{bmatrix}\\ &\qquad + \begin{bmatrix}D_GE & -D_GED_K\\ ED_KD_G & -ED_K\end{bmatrix}\begin{bmatrix}v\\r\end{bmatrix}\end{aligned}$$

By selecting specific columns of B and rows of C we can get a state-space realization for each of the four individual transfer functions in N.

If we look at the controllability and observability conditions for each of these cases we find the following:

$N(s)$	**Transfer function**	**# observable states**	**# controllable states**
$N_{11}(s)$	$S_\mathrm{O}(s)G(s)$	5	5
$N_{12}(s)$	$T_\mathrm{O}(s)$	5	4
$N_{21}(s)$	$-T_\mathrm{I}(s)$	4	5
$N_{12}(s)$	$S_\mathrm{I}(s)K(s)$	4	4

Only the $N_{11}(s)$ has 5 observable and controllable states. In each of the other cases the $s = 0$ state is either unobservable or uncontrollable or both. For the later three cases a minimal realization would have only the 4 states strictly inside the left-half plane.